If we think of the area under the graph of \( y = f(x) \),

\[
\int_a^b f(x) \, dx
\]

is the area under the graph of \( y = f(x) \), provided that the area makes sense.

(If with conventional area, \( \sum a_i \cdot f(x_i) \))

Here are many functions for which this doesn't work.

Get area as a suitable limit of Riemann sums. Ex: \( \int_1^3 \sqrt{x} \, dx \)

Ex: 5 equal length subintervals. Length \( \frac{b-a}{5} = \frac{2}{5} \)

Pick a point in each subinterval. Let's say left endpoints.

Intervals are \([1, 1 + \frac{2}{5}], [1 + \frac{2}{5}, 1 + \frac{4}{5}], \ldots, [1 + \frac{4}{5}, 1 + \frac{5}{5}]\)

Add up areas of red rectangles:

\[
\sum_{k=1}^{5} \left[ \sqrt{1 + \frac{2}{5}} \left( \frac{2}{5} \right) + \sqrt{1 + \frac{4}{5}} \left( \frac{2}{5} \right) + \cdots + \sqrt{1 + \frac{10}{5}} \left( \frac{2}{5} \right) \right]
\]

\[
\text{height of rectangle} = \frac{5}{\sqrt{\frac{2}{5}}}
\]

Area is approximated by Riemann sums. Recall the against: if \( v(t) \) is velocity at time \( t \), then distance particle goes between \( t = a \) and \( t = b \) is approximated by Riemann sums for \( \int_a^b v(t) \, dt \). Therefore, actual distance goes is \( \int_a^b v(t) \, dt \). We need to take

\[
\int_a^b v(t) \, dt < 0 \quad \text{if} \quad v(t) < 0 \quad \text{on} \quad [a, b],
\]

since particle is going backwards.
Fundamental Theorem of Calculus: If $f$ is continuous, then the function $F(x) = \int_a^x f$ is defined (the integrals exist), differentiable, and $F'(x) = f(x)$.

In particular, continuous functions always have antiderivatives, even if you can’t write down a formula for me with what we’ve said.

Conseq. 10 to find $\int_a^b f$ as discussed on prev. page, find some antiderivative $F_b$ and
then $\int_a^b f = F(b) - F(a)$.

So we can use everything you learned in Math 251.

\[
\int_1^3 \frac{1}{\sqrt{2x}} \, dx = \int_1^3 x^{-\frac{1}{2}} \, dx = \left[ \frac{x^{\frac{1}{2}}}{\frac{1}{2}} \right]_1^3 = \frac{2}{3} x^{\frac{3}{2}} \bigg|_1^3 = \frac{2}{3} (3)^{\frac{3}{2}} - \frac{2}{3} (1)^{\frac{3}{2}} = 2\sqrt{3} - \frac{2}{3} \text{ using } 3^{\frac{3}{2}} = 3\sqrt{3}.
\]

Note: If $f(x)$ is given by a formula, then $F(x)$ is given by a formula, but an antiderivative, often is not.

We spent time on methods for finding formulas for antiderivatives when they do exist

Tricky: writing backwards from diff. rules.

Five methods:

1) Immediate recognition: \[ \int 4x^2 \, dx = x^4 + C \]

\[ \int \cos(t) \, dt = \sin(t) + C \]

\[ \int \frac{1}{1+x^2} \, dx = \arctan(x) + C \]

\[ \int \sec^2(\theta) \, d\theta = \tan(\theta) + C \]

2) "Basic" combinations: sums, differences, scalar multiples, from $(f+g)' = f' + g'$, $c f' = cf'$

\[ \int (f+g) = \int f + \int g \] etc.

Exs: \[ \int x^4 \, dx = \int (\frac{1}{4}) 4x^4 \, dx = \frac{1}{4} x^4 + C \]

3) Substitution, "inverse" of chain rule.

\[ \int e^x \left( \frac{2}{1+x^2} \right) \, dx = \int e^x \, dx - 2 \int \frac{1}{1+x^2} \, dx = e^x - 2 \arctan(x) + C \]

Exs: \[ \int 5 \sin^4(x) \cos(x) \, dx = \int 5 u^4 u' \, du \] when $u(x) = \sin(x)$.

Conversion: \[ \int 5u^4 \, du = u^5 + C = \sin^5(x) + C. \]
Warning: If you don't have at least a constant multiple of \( x \) in \( x^2 \), it doesn't work.

If \( \int 5 \sin^4(x) \, dx \) this way, get \( \int 5 \sin^4(x) \, dx \),

Could put \( \int 5 \cos(x) \, du \)

correct, but still have both \( x \) and \( u \),

so can't do anything with \( x \).

It does not work to take \( v = \sin^3(x) \), since \( dv = 3 \sin^2(x) \cos(x) \, dx \).

Get \( \int 5 \cos(x) \, dx \)

bad since have both \( v \) and \( x \).

**Not a scalar multiple of \( dv \)**

Ex: Suppose \( F \) is an antiderivative of \( f(x) = e^{-x^2} \).

Find \( \int F(x) e^{-x^2} \, dx \)

\( u = F(x) \)

so \( du = F'(x) \, dx = e^{-x^2} \, dx \)

\( = \int u \, du = \frac{u^2}{2} + C = \frac{F(x)^2}{2} + C \).

Ex: \( \int \cos(2x) \, dx \)

\( u = 2x \) so \( du = 2 \, dx \)

We didn't quite see \( du \) but we see a scalar multiple of \( u \).

\( \frac{1}{2} \int \cos(2x) 2 \, dx = \frac{1}{2} \int \cos(2x) \, dx = \frac{1}{2} \sin(2x) + C = \frac{1}{2} \sin(2x) + C. \)

**Ok since \( 2 \) is a constant**

(4) Integration by parts — "inversion" of product rule.

Basic ex: \( \int x \cos(x) \, dx = x \sin(x) - \int 1 \cdot \sin(x) \, dx = x \sin(x) + \cos(x) + C \).

\( u \) \( (x) \) \( u \) \( \sin(x) \) \( v \) \( \cos(x) \) \( v \) \( (x) \)

It is easy that \( u \, dv = \frac{d}{dx} [u \cdot v(x)] - u \cdot v'(x) \), just to product rule.

**Principle:** Factor integral as \( u \cdot v'(x) \) in such a way that \( u \cdot v'(x) \)

is easier to integrate than \( u \cdot v(x) \) (and with \( v'(x) \) you can find).

\[ F(x) = \int x \cos(x^2) \, dx. \]

Then find \( \int x \cos(x^2) \, dx \).

Almost certainly can be done in terms of elementary functions and \( F \).

This one can be: \( \int x \cos(x^2) \cdot \cos(x) \, dx \).

(should not need \( F \))
Use substitution $u = \sin(t^2)$, so $du = 2t \cos(t^2)dt$, and $t \cos(t^2)dt = \frac{1}{2}du$.

So $\int \cos(\sin(t^2)) + \cos(t^2)dt = \frac{1}{2} \int \cos(u)du - \frac{1}{2} \sin(u) + C = \frac{1}{2} \sin(\sin(t^2)) + C$ and $\int_{b}^{a} \cos(\sin(t^2)) + \cos(t^2)dt = \frac{1}{2} \sin(\sin(t^2)) \bigg|_{b}^{a} = \frac{1}{2} \sin(\sin(b^2)) - \frac{1}{2} \sin(\sin(2))$.

**Extended example:** $\int \cos(\sin^2(t)) \cos(t)dt$. This now needs $F$.

If you assume it needs $F$, you want to substitute so as to get something like $\int \cos(u^2)du$. But $u = \sin(t)$ gives $du = 2 \sin(t) \cos(t)dt$.

The integral becomes $\int \frac{\cos(u)}{2 \sin(t)}du$ variables don't match, and $2 \sin(t)$ is not a constant, so this looks bad.

Instead take $u = \sin(t)$. So $du = \cos(t)dt$.

$\int \cos(\sin^2(t)) \cos(t)dt = \int \cos(u^2)du = F(u) + C = F(\sin(t)) + C$.

Warning: Can't find $\int \cos(\sin^2(t))dt$ this way: if put $u = \sin(t)$, then no $du$ in the integrand.

(5): algebraic trickery (added after end of review session no video)

**Examples**

$\int \frac{x^2 + 7}{x^2}dx = \int \left( x^{2/2} + x^{-3/2} \right) dx = \int x^{3/2}dx + 7 \int x^{-3/2}dx = -\frac{3}{2}x^{5/2} - \frac{14}{\sqrt{x}} + C$.

$\int \tan^2(x)dx = \int \sec^2(x) - 1 dx = \int \sec^2(x)dx - \int 1dx = \tan(x) - x + C$.

$\int \cot(x)dx = -\int \frac{\cos(x)}{\cos^2(x)}dx = -\int \frac{1}{u}du = -\ln|\cos(x)| + C$.

$u = \cos(x)$

$\int e^x \sec(x)dx = -\int e^{x-3x}dx + \int e^{x}dx$ use substitution $u = 2x$

$\int e^{x+3x}dx = \int e^{2x}dx + \int e^{3x}dx$

Parts, then $s$, last.