Ex (last time). A valley floor is shaped like the region between \( y = 81 - x^2 \) and the \( x \)-axis. The straight (south) side is along the ocean. The population density \( y \) miles inland is \( 100 - y \) people/square mile. What is the total population?

The population density is constant along any horizontal line.

(2.1 formula. \( 100 - y \) depends only on \( y \), and thus on the line on which \( y \) is constant.)

\( dx \) should be small, then \( dy \) is drawn. The population density is approximately constant in entire rectangle, because \( y \) does not vary very much. Thus rectangle has length (horizontal) \( 2x = 2(81 - y)^{1/2} \), so has area \( 2x \Delta y = 2(81 - y)^{1/2} \Delta y \).

The \( x \)'s don't match. To be useful, these must match.

This is why \( x \) must be replaced by an expression involving only \( y \).

The rectangle has area \( 2(81 - y)^{1/2} \Delta y \). (in square miles).

How many people (approximately) live in this rectangle?

The pop density in rectangle is \( 1 \cdot 100 - y \) people/square mile. (since \( \Delta y \) is small, so \( \frac{dx}{dy} \approx 0 \)).

So its population is \( \int_{a}^{b} (100 - y) \cdot 2(81 - y)^{1/2} \Delta y \).

Imagine the valley floor being divided into many rectangles like this. Add their populations, getting a Riemann sum for some integral.

The integral is: \( \int_{100}^{81} (100 - y) \cdot 2(81 - y)^{1/2} \, dy \).

\( a \) and \( b \) are supposed to be values of \( y \), so \( a = 0 \) and \( b = 81 \).

Find \( a \) later.

Answer will be \( \int_{0}^{81} 2(100 - y) (81 - y)^{1/2} \, dy \).


\[ \text{Suppose the pop density \( f(x, y) \)} \text{ were instead } 70 + x \text{ people per square mile.} \]

Would need to use vertically oriented rectangles, to make the pop density roughly constant in each rectangle. Do this at home. Should get:

\[ \int_{-3}^{3} (70 + x) (81 - x^2) \, dx \] for total population.

\[ \text{Pop density} \frac{\text{height}}{\text{width}} \text{ of the rectangles used.} \]
Ex: An unbalanced marble has radius 1 cm and, when the heavy side is down, density \( \frac{3-x}{g/cm^2} \) at height \( x \) cm above its center. What is its mass?

- \( x = 1 \)
- \( x = 0 \)

At height \( x \) <- intersection of a horizontal plane with the sphere.

- \( x = -1 \) Use horizontal because density is constant on this plane.

Make the plane to make a cylinder of height \( \Delta x \).

The cylinder has radius \( \sqrt{1-x^2} \) cm (radius of the red circle). (Note: use \( x \) as section.)

Center of sphere

Center of red circle

Right triangle

The cylinder has radius \( \sqrt{1-x^2} \) cm, height \( \Delta x \), and density about \( 3-x \) g/cm².

Its volume is: \( \pi \left( \sqrt{1-x^2} \right)^2 \Delta x \). So mass is \( \int_{-1}^{1} \pi \left( \sqrt{1-x^2} \right)^2 \Delta x \) (3-x).

This is the mass of the green cylinder.

The total mass will be approximated by a sum of terms like this, so an approximating Riemann sum is: \( \sum_{k=1}^{n} \pi \left( \sqrt{1-x_k^2} \right)^2 (3-x_k) \Delta x \), and the integral giving the true mass is:

\[
\int_{-1}^{1} \pi \left( \sqrt{1-x^2} \right)^2 (3-x) \, dx = \int_{-1}^{1} \pi (1-x^2) (3-x) \, dx
\]

If left this out, would get the volume of the sphere.