1. (10 points.) Consider the region between the graph of $y = \sin(x^2)$, the $x$-axis, $x = \sqrt{\pi}$, and $x = \sqrt{2\pi}$. It is rotated about the $y$-axis. Find the volume of the resulting solid.

Solution: Here are three pictures. The solid lies above the surfaces shown at the left, in the middle is the graph of $y = \sin(x^2)$ on $[-3, 3]$, and on the right is the outline of the region actually being rotated.

We use cylindrical shells. For $0 \leq x \leq 1$, the cylindrical shell at this value of $x$ and with thickness $\Delta x$ has radius $x$. Since $\sin(x^2) \leq 0$ when $x$ is in $[\sqrt{\pi}, \sqrt{2\pi}]$, its height is $0 - \sin(x^2) = -\sin(x^2)$. So its volume is approximately $2\pi x(-\sin(x^2)) \Delta x$. This leads to the integral

$$\int_{\sqrt{\pi}}^{\sqrt{2\pi}} 2\pi x(-\sin(x^2)) \, dx.$$ 

We do this integral with the substitution $u = x^2$, so $du = 2x \, dx$. Thus

$$\int 2\pi x(-\sin(x^2)) \, dx = -\int \pi \sin(u) \, du = \pi \cos(u) + C = \pi \cos(x^2) + C,$$

and

$$\int_{\sqrt{\pi}}^{\sqrt{2\pi}} 2\pi x(-\sin(x^2)) \, dx = \pi \cos(x^2) \bigg|_{\sqrt{\pi}}^{\sqrt{2\pi}} = \pi \cos(2\pi) - \pi \cos(\pi) = \pi - (-\pi) = 2\pi.$$

2. (5 points.) The planet Yuggxth is a sphere with radius 6000 kilometers. (It has no mountains, no valleys, and no oceans.) The density of its atmosphere at height $h$ kilometers above the surface, measured in millions of metric tons per cubic kilometer, is $1.2 \exp(-0.13h)$. Write down an integral which represents the total mass of the part of the atmosphere of Yuggxth which is between the surface of the planet and 100 kilometers above the surface. Be sure to take account of the fact that the planet is not flat. Include an explanation. Don’t try to evaluate the integral.

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Solution: For $0 \leq h \leq 100$ and a very small number $\Delta h$, consider the part of the atmosphere of Yuggxth which is between $h$ and $\Delta h$ kilometers above the surface. This is a thin spherical shell with inner radius $6000 + h$ kilometers. The surface area of a sphere with radius $6000 + h$ kilometers is $4\pi(6000 + h)^2$ square kilometers, so our thin shell has volume approximately $4\pi(6000 + h)^2 \Delta h$ cubic kilometers. The density is nearly the same throughout this shell, so the part of the atmosphere in this shell has mass $4\pi(6000 + h)^2 \Delta h \cdot 1.2 \exp(-0.13h)$ metric tons.

Now consider a subdivision of $[0, 100]$ into $n$ intervals, each of length $\Delta h = 100/n$. (We don’t have to use equal length intervals; this is just for convenience.) For $k = 1, 2, \ldots, n$ let $h_k$ be the left endpoint of the $k$-th interval. The total mass of the atmosphere between 0 and 100 kilometers above the surface is then approximately

$$
\sum_{k=1}^{n} 4\pi(6000 + h_k)^2 \cdot 1.2 \exp(-0.13h_k) \Delta h.
$$

This looks like a Riemann sum for \( \int_{0}^{100} 4\pi(6000 + h)^2 \cdot 1.2 \exp(-0.13h) \, dh \). So the total mass of the part of the atmosphere which is between the surface and 100 kilometers above the surface is

$$
\int_{0}^{100} 4.8\pi(6000 + h)^2 \exp(-0.13h) \, dh
$$

metric tons.

(To get full credit, a solution does not have to contain an explicit Riemann sum as in (1) above. But the reasoning behind the expression $4\pi(6000 + h)^2 \cdot 1.2 \exp(-0.13h) \Delta h$ or $4\pi(6000 + h)^2 \cdot 1.2 \exp(-0.13h) \, dh$ must be made clear.)

3. (15 points.) The city of Megalopolis is circular and has a radius of 10 miles. Its population density $r$ miles from the city center is \( \left(\frac{1}{\pi}\right)(30,000 - 30r^3) \) people per square mile. Find the average distance residents of Megalopolis live from the center of the city. Include an explanation.

Solution: Consider a thin annulus (ring shaped area) at distance $r$ miles from the center of the city, and with width $\Delta r$. The circumference of a circle with radius $r$ is $2\pi r$, so this annulus has area approximately $2\pi r \Delta r$. (The exact area is

$$
\pi(r + \Delta r)^2 - \pi r^2 = 2\pi r \Delta r + \pi (Dtr)^2
$$

Since $\Delta r$ is supposed to be very small, we can drop terms involving $(Dtr)^2$.) The number of people living in this ring is about

$$
\left(\frac{1}{\pi}\right)(30,000 - 30r^3) \cdot 2\pi r \Delta r.
$$

Since they are all at distance approximately $r$ from the center of the city, the contribution to the numerator in the fraction representing the average distance is

$$
\left(\frac{1}{\pi}\right)(30,000 - 30r^3) \cdot 2\pi r \Delta r \cdot r = \left(\frac{1}{\pi}\right)(30,000 - 30r^3) \cdot 2\pi r^2 \Delta r.
$$

Now consider a subdivision of $[0, 10]$ into $n$ intervals, each of length $\Delta r = 10/n$. (We don’t have to use equal length intervals; this is just for convenience.) For $k = 1, 2, \ldots, n$ let $r_k$ be the left endpoint of the $k$-th interval. The total population is approximately

$$
\sum_{k=1}^{n} \left(\frac{1}{\pi}\right)(30,000 - 30r_k^3) \cdot 2\pi r_k \Delta r.
$$
This looks like a Riemann sum for $\int_0^{10} \left( \frac{1}{\pi} \right) (30,000 - 30r^3) \cdot 2\pi r \, dr$. The actual total population is thus
\[
\int_0^{10} \left( \frac{1}{\pi} \right) (30,000 - 30r^3) \cdot 2\pi r \, dr = \int_0^{10} (60,000 r - 60r^4) \, dr
\]
\[
= (30,000 r^2 - 12r^5) \bigg|_0^{10} = 30,000 \cdot 10^2 - 12 \cdot 10^5 = 3,000,000 - 1,200,000 = 1,800,000.
\]

The average distance of the population from the center of the city is now approximately
\[
\frac{\sum_{k=1}^{n} \left( \frac{1}{\pi} \right) (30,000 - 30r_k^3) \cdot 2\pi r_k^2 \Delta r}{1,800,000}.
\]

This looks like a Riemann sum for $\frac{1}{1,800,000} \int_0^{10} \left( \frac{1}{\pi} \right) (30,000 - 30r^3) \cdot 2\pi r^2 \, dr$. The actual average distance is thus
\[
\frac{1}{1,800,000} \int_0^{10} \left( \frac{1}{\pi} \right) (30,000 - 30r^3) \cdot 2\pi r^2 \, dr = \frac{1}{1,800,000} \int_0^{10} (60,000 r^2 - 60r^5) \, dr
\]
\[
= \left( \frac{1}{1,800,000} \right) (20,000 r^3 - 10r^6) \bigg|_0^{10}
\]
\[
= \left( \frac{1}{1,800,000} \right) (20,000 \cdot 10^3 - 10 \cdot 10^6)
\]
\[
= \left( \frac{1}{1,800,000} \right) (20,000,000 - 10,000,000) = \frac{50}{9}.
\]

Thus, on average, the residents of Megalopolis live $50/9$ miles from the center of the city. (The units are required.)

(To get full credit, a solution does not have to contain an explicit Riemann sum as in (2) and (3) above. But the reasoning behind the expressions
\[
\left( \frac{1}{\pi} \right) (30,000 - 30r^3) \cdot 2\pi r \Delta r \quad \text{or} \quad \left( \frac{1}{\pi} \right) (30,000 - 30r^3) \cdot 2\pi r \, dr,
\]
and
\[
\left( \frac{1}{\pi} \right) (30,000 - 30r^3) \cdot 2\pi r^2 \Delta r \quad \text{or} \quad \left( \frac{1}{\pi} \right) (30,000 - 30r^3) \cdot 2\pi r^2 \, dr
\]
must be made clear.)

Also, one doesn’t really need to write out the calculations separately; one can go to
\[
\int_0^{10} \left( \frac{1}{\pi} \right) (30,000 - 30r^3) \cdot 2\pi r^2 \, dr
\]
\[
\int_0^{10} \left( \frac{1}{\pi} \right) (30,000 - 30r^3) \cdot 2\pi r \, dr
\]
directly.)

4. (10 points.) On 1 January 1970, Wang’s Widgets Inc. was selling widgets at the rate of 200,000 widgets per year. Its rate of sales over the next three decades is modelled as 200,000 $e^{0.02t}$ widgets per year $t$ years after 1970. According to this model, how many widgets did Wang’s Widgets Inc. sell during the period from 1 January 1970 through 31 December 1999 (a total of 30 years)? Include an explanation.

Solution: Consider the time interval from $t$ to $t + \Delta t$, with $t$ in $[0, 3]$ and $\Delta t$ very small. Since the time interval is very short, the rate of sales of widgets was about the same (namely 200,000 $e^{0.02t}$ widgets per year) over the interval, so the total number of widgets sold was $200,000 e^{0.02t} \Delta t$. 
5. (10 points.) A 5 meter chain is lying on the floor of a room with a flat floor and a 10 meter high ceiling. It has a hook at one end. Its linear density $x$ meters from the end with the hook is $4 + 2 \cos(\pi x)$ kilograms per meter. How much work is required to lift the chain so that it hangs straight down from the ceiling, with the hook at the ceiling?

Solution: Consider a short segment of the chain, with length $\Delta x$ meters, and located $x$ meters from the end with the hook. Its linear density is approximately $4 + 2 \cos(\pi x)$ kilograms per meter, so its mass is approximately $(4 + 2 \cos(\pi x)) \Delta x$ kilograms. Its final position is $x$ meters below the ceiling, so it must be lifted $10 - x$ meters, against a downwards acceleration of $g = 9.8 \text{ m/sec}^2$. Therefore the work in Newtons is $$9.8(10 - x)(4 + 2 \cos(\pi x)) \Delta x.$$ 

By considering Riemann sums, we see that we need:

$$\int_0^5 9.8(10 - x)(4 + 2 \cos(\pi x)) \, dx = \int_0^5 (9.8 \cdot 10 \cdot 4 + 9.8 \cdot 10 \cdot 2 \cos(\pi x) - 9.8 \cdot x \cdot 4 - 9.8 \cdot x \cdot 2 \cos(\pi x)) \, dx$$

$$= \int_0^5 4 \cdot 98 \, dx + \int_0^5 98 \cdot 2 \cos(\pi x) \, dx - 9.8 \int_0^5 4 \cdot x \, dx - 9.8 \int_0^5 2x \cos(\pi x) \, dx.$$ 

We have

$$\int_0^5 4 \cdot 98 \, dx = 20 \cdot 98 = 196,$$

$$9.8 \int_0^5 4 \cdot x \, dx = (9.8 \cdot 2x^2) \bigg|_0^5 = 9.8 \cdot 2 \cdot 5^2 = 49,$$
and
\[ \int_{0}^{5} 98 \cdot 2 \cos(\pi x) \, dx = \left. \left( \frac{98 \cdot 2}{\pi} \sin(\pi x) \right) \right|_{0}^{5} = \frac{98 \cdot 2}{\pi} (\sin(5\pi) - \sin(0)) = 0. \]

The last integral is done by parts. Take \( u(x) = 2x, \) \( v'(x) = \cos(\pi x), \) \( u'(x) = 2, \) and \( v(x) = \frac{1}{\pi} \sin(\pi x), \) to get
\[
\int_{0}^{5} 2x \cos(\pi x) \, dx = \left. \frac{2}{\pi} x \sin(\pi x) \right|_{0}^{5} - \int_{0}^{5} \frac{2}{\pi^2} \sin(\pi x) \, dx
\]
\[
= \frac{2}{\pi} \sin(5\pi) - 0 + \frac{2}{\pi^2} \cos(5\pi) + \frac{2}{\pi^2} \cos(0)
\]
\[
= 0 - 0 + \frac{2}{\pi^2} (-1) - \frac{2}{\pi^2} (1) = -\frac{4}{\pi^2}.
\]

Thus, the total work is \( 196 + 49 + \frac{4}{\pi^2} = 147 + \frac{4}{\pi^2} \) Newton-meters.

(The units are required.)