Recall (justification in 2nd part last Sec.) Average value of \( f \) on \([a, b]\) is \( \frac{1}{b-a} \int_a^b f(x) \, dx \)

Example: Avg. value of \( e^x \) on \([3, 8]\) is \( \frac{1}{5} \int_3^8 e^x \, dx = \frac{1}{5} e^8 - e^3 = \frac{1}{5} \) 

Warning: \( \int_0^\infty \cos(x) \, dx \) converges (1)

To find an average value on \([a, b]\), take \( \frac{1}{b-a} \int_a^b f(x) \, dx \). If it doesn't exist.

Another solid of revolution: Region between \( f(x) = y = x^3 + x \) and \( y = 2x^3 \), to right of \( y = x^3 + x \). 

Rotate about \( x = -3 \). Need to know where curves cross.

Solve \( x^3 + x = 2x^3 \), which is \( x^2 - x = 0 \), so \( x(x-1)(x+1) = 0 \), so \( x = 0, 1, \) or \(-1\).

Plugging in these points: \( 0, 2, -2 \).

This is the eqn \( f(1) = f(2) \).

Since \( x = 1 \) is a solution, must have \( f(1) = f(1) \). Here the value is 2.

Use shells. 

A small shell at position \( x \) has height \( f(x) - f(1) = x^3 - 2x^3 = x - x^2 \), thickness \( dx \) (became \( dx \) in the integral). 

Circumference \( 2 \pi (x + 3) \), so volume about \( 2 \pi (x + 3)(x - x^2) \). 

Total volume of the solid is \( \int_0^1 2 \pi (x + 3)(x - x^2) \, dx \) (0 and 1 are the value of \( x \) at the edges).

To do integral: multiply out.

When you look at what happens between \( x \) and \( x + \Delta x \) (volume, or something else), linearize as a function of \( \Delta x \) (or expand in pieces of \( \Delta x \). If there is a term without \( \Delta x \), not an integration problem. Otherwise, when let \( \Delta x \rightarrow 0 \), anything with (\( \Delta x \))^2, \( (\Delta x)^3 \), etc. will go to 0.

For wiscons, get the exact volume. For shells, get volume up to something with \( (\Delta x)^2 \).

Heuristically, you are adding \( \Delta x \) term and letting \( \Delta x \rightarrow 0 \).

The ones of size \( \Delta x \) contribute something substantial of size multiple of \( \Delta x \), so zero in the limit.
Work (physics) Suppose you move a particle distance $d$ against a constant force $F$. The work done is $Fd$.

Example: lifting an object of mass $2$ kg to a height of $6$ m, against gravity, $g = 9.8 \text{ m/s}^2$. Force is $F = 2 \times 9.8$ N, so work is $(2)(9.8)$ kg$\cdot$m$^2$/sec$^2$. $= 19.6$ N$\cdot$m (or $19.6$ J, since work is done in joules).

Ex. You have two particles of masses 3 and 5 kg, connected by a massless string, of length 4 m. The heavier of mass 5 kg is hung from the ceiling on the fifth floor of the gas giant planet Gleep. What is the time due to gravity? $g = 6$ m/sec$^2$.

Left the heavier bill 10 m off the ground. How much work is done?

Heavier bill is lifted 10 m against a force of $(5 \text{ kg})(6 \text{ m/sec}^2)$, so work is $(5)(5)(6)$ N$\cdot$m $= 150$ N$\cdot$m.

Lighter bill is lifted 6 m against a force of $(3 \text{ kg})(6 \text{ m/sec}^2)$, so work is $(3)(3)(6)$ N$\cdot$m $= 54$ N$\cdot$m.

Ex. A chain of length 2 m, linear density 10 kg/m, is in free fall. At 2 m height, the ends are lifted 2 m higher, to hang down from 20 meters off the ground.

\[ \text{Distance from top} \]

\[ \text{Work done is about } (18+x)(40\Delta x) \text{ by distance from top.} \]

\[ \int_0^2 (18+x)(40\Delta x) \text{ in Newtons.} \]

\[ \text{At positions } 0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3} \text{ moved against force } (\frac{1}{3})(10)(10) \text{ and moves distance } 18 + \frac{2}{3} \text{ (approximately).} \]

\[ \text{Bottom end moves } 18 + \frac{2}{3}, \text{ top end moves } 18 + \frac{2}{3} + 18 + \frac{2}{3} + \Delta x. \]

\[ \text{Add them: } \frac{1}{3}(10)(10)(18) + \frac{1}{3}(10)(10)(18 + \frac{1}{3}) + \frac{1}{3}(10)(10)(18 + \frac{2}{3}) + \ldots \]

\[ \Rightarrow \int_0^2 (10)(10)(18+x)dx. \]

This is the Riemann sum for 6 equal length subintervals using left endpoints.