Example Consider a spring: \[ \text{mass } m \quad \begin{array}{c} \text{free at the edge of } x \\ \text{exerts force } -k\lambda x \end{array} \]

Let \( x \) be the displacement from the natural length. According to Hooke's Law, for a small

\( x \) the spring exerts a restoring force \(-kx\) for some constant \( k > 0 \) depending on the spring

(let us ignore non-linear effects since physics we'll treat \( k \) constant). Newton's Law says force equals mass times acceleration.

Assume no friction. One gets: \[ m \ddot{x}(t) = -kx(t). \]

This is a differential eqn with unspecified constant \( m \) and \( k \). Rewrite slightly: \[ \ddot{x}(t) = -\frac{k}{m} x(t). \]

Let's try \( x(t) = \sin \left( \sqrt{\frac{k}{m}} t \right) \) Is \( x(t) \) a solution?\[ \dot{x}(t) = \sqrt{\frac{k}{m}} \cos \left( \sqrt{\frac{k}{m}} t \right) \cdot \sqrt{\frac{k}{m}} = \sqrt{\frac{k}{m}} \sin \left( \sqrt{\frac{k}{m}} t \right) \]

Is \( \ddot{x}(t) = -\frac{k}{m} x(t) \) for all \( t \)?\[ \ddot{x}(t) = \sqrt{\frac{k}{m}} \frac{k}{m} \sin \left( \sqrt{\frac{k}{m}} t \right) \cdot \sqrt{\frac{k}{m}} \sin \left( \sqrt{\frac{k}{m}} t \right) = -\frac{k}{m} x(t) \]

Is \( x(t) = \sin \left( \sqrt{\frac{k}{m}} t \right) \) a solution?\[ x(t) = \frac{k}{m} \sin \left( \sqrt{\frac{k}{m}} t \right) \quad \text{and} \quad \dot{x}(t) = -\left( \frac{k}{m} \right)^2 \cos \left( \sqrt{\frac{k}{m}} t \right). \]

Is this the same as \(-\frac{k}{m} x(t)\)? True if \( \frac{k}{m} = 1 \). Since \( \tan \left( \frac{k}{m} t \right) = \frac{k}{m} \), also if \( \frac{k}{m} = 0 \).

Otherwise, no: \( x_0(t) = \cos \left( \sqrt{\frac{k}{m}} t \right) \), then \( \dot{x}_0(t) = -\left( \frac{k}{m} \right)^2 \) but \(-\frac{k}{m} x(t) = -\frac{k}{m} \cos \left( \sqrt{\frac{k}{m}} t \right) = \not= \dot{x}_0(t) \).

How about \( x_0(t) = \cos \left( \sqrt{\frac{k}{m}} t \right) \)? Is this a solution? Yes, similar to \( x(t) = \sin \left( \sqrt{\frac{k}{m}} t \right) \).

The real reason trig functions appear so much in the course you have taken is that they are used

in solving to differential equations like this one. There is no circle or angle in sight!

Motion of a mass on a spring looks like a special solution, but in fact the same eqn occurs

for small displacements from almost any stable equilibrium point in physics or anywhere else.

Notation: For eqn \( \dot{y}(x) = xy(x) \), we often see \( \frac{dy}{dx} = xy \) or \( y' = xy \)

you are supposed to realize that the independent variable is called \( x \).
You have already seen differential eqns, such as: \( F(x) = x \sin(x), \) or \( F(x) = xe^{-x} \).

For a first order eqn, you can solve it easily, for ex, for the second, \( f(x) = -\frac{1}{2}e^{-x^2} + C \).

For a second order eqn, say \( f''(x) = \) const, you have two arbitrary constants. Here integrate twice: \( f'(x) = \frac{7}{2}x^2 + c_1x + c_2 \) for constants \( c_1 \) and \( c_2 \).

In many eqns from models, the variable is time, but not always.

Example For which real numbers \( r \) does the function \( f(x) = e^{rx} \) satisfy \( 2f'' + f' + f = 0 \)?

How to do this? We need: \( f(x) = e^{rx} \), \( f'(x) = re^{rx} \), \( f''(x) = r^2e^{rx} \). So the eqn becomes:

\[
2r^2e^{rx} + re^{rx} - e^{rx} = 0 \quad \text{so} \quad (2r^2 + r - 1)e^{rx} = 0. \]

Supposed to be true for all \( x \), so \( 2r^2 + r - 1 = 0 \). Thus has solution \( r = \frac{1}{2} \) and \( r = -1 \). Check: both \( e^{\frac{1}{2}x} \) and \( e^x \) are solutions.

For Find \( b \) such that \( f(x) = x^3 + 2x^2 \) satisfies the eqn: \( f(x) - 3f'(x) = bx^2 \).

How to do this? Find \( f'(x) = 3x^2 + 4x \). The eqn becomes: \( (3x^2 + 4x) - 3(6x + 2x^2) = bx^2 \).

Simplify: \( 3x^2 + 4x^2 - 3x^2 - 6x^2 = bx^2 \). Left with \( -2x^2 = bx^2 \). Supposed be true for all \( x \), so \( b \) must be \(-2\).

Example Consider diff eqn \( y'(x) = \frac{1}{10} y(x) \left( 4 - y(x) \right) \) for \( t \in (-\infty, \infty) \)

\[
m = \frac{1}{10} y(4 - 1) \quad \text{Assume we have a solution } \ y(t) \quad \text{and that } y(1) = 3.
\]

(1) What is \( y'(1) \)?

(2) Is \( y(t) \) increasing or decreasing near \( t = 11 \)?

(1) To find \( y'(1) \): \( y'(1) = \frac{1}{10} y(1) (4 - y(1)) = \frac{1}{10} (3)(4 - 3) = \frac{3}{10} \).

(2) \( y(11) = 0 \) so \( y \) should be increasing near \( t = 11 \).

Suppose instead \( y(11) = 6 \). Then \( y'(11) = \frac{1}{10} y(11) (4 - y(11)) = \frac{1}{10} (6)(4 - 6) = -\frac{12}{10} \).

So \( y \) should be decreasing near \( t = 11 \).

Back to \( y(11) = 3 \). Approximate \( y(11.1) \) using the linear approximation.

Also if \( y(11) = 6 \) instead.