Logistic equation: \( P'(t) = kP(M-P) \) for constants \( k, M > 0 \). (model of population growth with fixed carrying capacity of the environment; \( M \) is the growth rate in absence of constant.)

1. There are constant solutions \( P(t) = 0 \) for all \( t \), and \( P(t) = M \) for all \( t \). (Disc will be lost in 50% of variables, since will divide by \( P(t) (1 - P(t)) \).)

2. For the other cases, separate variables, \( \frac{dP}{dt} = kP(1 - \frac{P}{M}) \). Rearrange slightly: \( \frac{dP}{P(M-P)} = k\,dt \).

Solve: \( \frac{M}{P(M-P)} \, dP = k\,dt \). Integrate: \( \int \frac{M}{P(M-P)} \, dP = \int k\,dt = kt + C_0 \) for some constant \( C_0 \).

Method: partial fractions. Let \( \frac{M}{P(M-P)} \) be a rational function for which we previously methodically apply use partial fractions. \( \frac{M}{P(M-P)} = \frac{a}{P} + \frac{b}{M-P} \), use to find \( a \) and \( b \).

RHS is \( \frac{a(M-P) + bP}{P(M-P)} \).

We want \( M = aM + (b-M)a \) for all values of \( P \). (Remember: \( M \) is a constant.)

Now it's easy: \( b - a = 0 \), so \( a = 1 \). Rearrange \( b = 1 \). Therefore \( \frac{M}{P(M-P)} = \frac{1}{P} + \frac{1}{M-P} \).

\( \int \frac{M}{P(M-P)} \, dP = \int \frac{1}{P} \, dP + \int \frac{1}{M-P} \, dP = \ln(|P|) - \ln(|M-P|) + C_1 \).

\( P \) does not appear on the left, so its antiderivative is zero.

Two ways. (1) equate coefficients of the powers of \( P \).

(2) Plug in several values of \( P \); here, for example, \( P = 0 \) and \( P = 1 \) are good choices.

Put together, and remember only need one constant:

\( \ln(|P|) - \ln(|M-P|) = kt + C_0 \). Rewrite: \( \ln\left(\frac{P}{M-P}\right) = kt + C_0 \).

Exponentiate: \( \frac{P}{M-P} = e^{kt+C_0} = e^{C_0}e^{kt} \).

Since \( P = 0, P = M \) (constant tend) we substitute, no solution curve can cross \( P = 0 \) or \( P = M \).

Rearrange \( \frac{P}{M-P} \) always has the same sign. So get solutions \( \frac{P}{M-P} = e^{C_0}e^{kt} \) and \( \frac{P}{M-P} = -e^{C_0}e^{kt} \).

Now take \( r = e^{C_0} \) in first case, \( r = -e^{C_0} \) in second case: \( \frac{P}{M-P} = r e^{kt} \) in both cases.

(r is another constant)
\[ M = \frac{P}{P - 1} \quad \text{for the constant } A = \frac{m}{1 + Ae^{-kt}}. \]

**Ex.** Suppose carrying capacity is 100, \( P(0) = 2 \) and \( P(1) = 4 \). What is \( k \)?

To set it up: \[ 2 = P(0) = \frac{100}{1 + Ae^{-k(0)}} \quad \text{and} \quad 4 = P(1) = \frac{100}{1 + Ae^{-k(1)}}. \]

There are two cases in two dimensions: solve. (The first one says \( 2 = \frac{100}{1 + A} \).)

**Ex.** of setting up an eqn.

A large chamber contains a mixture of nitrogen + oxygen. Chamber is 1000 m\(^3\).

It contains some animals which consume oxygen at rate of 2 m\(^3\)/day. A mixture of 20% oxygen and 80% nitrogen enters the bottom at 40 m\(^3\)/day, air in chamber is well mixed, and leaves top at 40 m\(^3\)/day. Set up differential eqn. for amount of oxygen in chamber assuming it starts at 80% N\(_2\), 20% O\(_2\). What is limiting value of the amount of oxygen?

Need to define variables and functions. It will be time, measured in days, since starting time.

Let \( V(t) \) be the volume of oxygen in the chamber at time \( t \), in m\(^3\).

There are three contributions to \( V(t) \):

1. air entering chamber, oxygen consumed, air leaving.
2. \( 40 \) m\(^3\)/day, of which 20% is oxygen, so \((0.2)(40) = 8\) m\(^3\)/day oxygen.
3. \(-2\) m\(^3\)/day.

(3) 40 m\(^3\)/day of air is leaving, but how much is oxygen? Total amount of O\(_2\) is \( V(t) \).

That is how much there is in 1000 m\(^3\). So, assuming well mixed, there is \( \frac{40}{1000} V(t) \) m\(^3\) of oxygen in 40 m\(^3\). So the contribution here is \(-\frac{40}{1000} V(t)\).

Putting these terms together: \( V'(t) = 8 - 2 - \frac{40}{1000} V(t) = 6 - \frac{1}{25} V(t) \).
Solve by separation of variables: \[ \frac{V'(t)}{6 - \frac{1}{25} V(t)} = t, \]

Eventually get: \[ V(t) = 150 + Ce^{-t/25} \]
for some constant \( C \), which you can find using \( V(0) = (0.2)(1000) = 200 \).

In any case, \( \lim_{t \to \infty} V(t) = 150 \) \( \left( \text{m}^3 \text{ of } O_2 \right) \).

\[ \int_0^t \frac{dV}{6 - \frac{1}{25} V} = \int dt \]

LHS is \(-25 \ln (6 - \frac{1}{25} V) + C_1\)