Example: A trough in the plant Kyung has a cross section in the shape of the graph of \( y = \frac{1}{4} x^4 \) (width in meters). It is 40 m long, filled to a depth of 4 meters with water with a density of 1000 kg/m³. On this plant, \( g = 5 \) m/s².

1. What is the force on one end of the trough?
2. The water was put there by being taken from a lake, 3 m below the bottom of the trough. How much work was needed?
3. At some time, somebody starts pumping salt water, 2 kg salt/m³ into the trough at 0.01 m³/sec. The water is well mixed and overflows at same rate. Find a formula for the amount of salt in trough as a function of time.

\[ y = \frac{1}{4} x^4 \]

**Values of \( x \) at the point "?"**

Depth is 4, so \( \frac{1}{4} x^4 = \frac{1}{4} x^4 \), which has solution \( x = \pm 2 \).

Use horizontal rectangle, because the force is not area is nearly constant, making it easy to find the total force.

The rectangle: area times pressure.

Area is \( 2 \times 4 h \), with \( x \) as shown: \( h = \frac{1}{4} x^4 \), so \( x = (4h)^{1/4} \). So area is \( 2 \times (4h)^{1/4} \) dh. Need pressure. Think of one square meter (think of as flat).

Force exerted by water above it? First how much water? Second, what is the mass? Third, what force does it exert?

\[ \text{mass is newtons} = 1000 \text{ kg/m}^3 \times (4-h) \text{ m}^3 \text{ which is } 1000 (4-h) \text{ kg. Force is newtons.} \]

\[ 1000 (4-h) \text{ kg/m}^3 \times 5 \text{ m/sec} \text{ or } 500 (4-h) \text{ kg/m/sec}^2 \text{. This is the force on one square meter. So pressure is } 5000 (4-h) \text{ N/m}^2. \]

So the force on the green rectangle is \( 2 \times (4h)^{1/4} \) dh \( \times 5000 (4-h) \) = \( 10,000 (4-h) \times (4h)^{1/4} \) dh.

\[ \text{Total force is } \int_0^4 10,000 (4-h) (4h)^{1/4} \text{ dh} \text{ (integrates from 0 to 4).} \]

\[ \text{How to do it?} \]

\[ \int_0^4 10,000 (4-h) (2h)^{1/4} \text{ dh} = \int_0^4 40,000 \sqrt{2} \text{ h}^{1/4} \text{ dh} - \int_0^4 10,000 \text{ h}^{5/4} \text{ dh} \]

\( (\text{limit rest.}) \)
(2) The water was put here by being taken from a lake, 3 m below the bottom of the trough. How much work was needed?

Still want to use a horizontal rectangle, since the work involved per unit mass (or unit volume) is nearly constant in this rectangle. So easy to write a formula for dh.

We still have area of \( 2 \cdot (4h)^{\frac{1}{4}} \Delta h \). Total volume of water with height between \( h \) and \( h + \Delta h \) is \( (40) \cdot 2 \cdot (4h)^{\frac{1}{4}} \Delta h = 80(4h)^{\frac{1}{4}} \Delta h \). Total mass of the water here is \( m = 1000 \cdot \frac{80(4h)^{\frac{1}{4}} \Delta h}{1 \text{ m}^3} \) kg.

Density, \( \frac{\text{kg}}{\text{m}^3} \) \Rightarrow \text{Volume, m}^3 \Rightarrow \text{Mass, kg} \)

Work involved to lift this is \( \frac{(\text{mass})(\text{distance})(\text{acceleration})}{\text{force}} \), which is \((\text{friction})(\text{force})\). We have the \( 3\text{th} \) \( \frac{5}{\text{m/\text{s}}} \).

\( \frac{80,000(4h)^{\frac{1}{4}}(3\text{th})(5\Delta h)}{2\text{th}} \). So total work is \( \int_0^4 80,000(4h)^{\frac{1}{4}}(3\text{th}) \cdot 5 \text{ dh} \).

Omit rest.

(3) At some time, somebody starts pumping salt water, 2 kg salt/m³ into the trough at 0.01 m³/sec. The water is well mixed, and overflows at same rate. Find a formula for the amount of salt in the trough as a function of time.
Just an exercise:

\[ V = \int_{-2}^{2} 40 \left( 4 - \frac{1}{4} x^4 \right) \, dx \] 

(And calculate);

Let \( S(t) \) be mass of salt at time \( t \) after start.

\[ S(t) = 0.02 - \frac{0.1 \sqrt{S(t)}}{\sqrt{x}} \]

\( 0.1 \) is salt/unit volume in trough

\( (2 \text{ kg/m}^3)(0.01 \text{ m}^3/\text{sec}) \)

\( 0.01 \) is volume/unit time of overflow.