Let $f$ be a function, let $a$ be a real number, and suppose in each case that enough of the derivatives
\[ f(a), f'(a), f''(a), f'''(a), \ldots, f^{(n)}(a), \ldots, \]
exist. We refer to $f(a)$ as the zeroth derivative. The expression $f^{(n)}(a)$ is the $n$-th derivative.

Recall: The $n$-th degree Taylor polynomial for $f$ at $a$ (or “centered at $a$”) is the polynomial $p_n(x)$ of degree at most $n$ such that
\[
p_n(a) = f(a), \quad p'_n(a) = f'(a), \quad p''_n(a) = f''(a),
\]
\[
p'''_n(a) = f'''(a), \quad \ldots, \quad \text{and} \quad p^{(n)}_n(a) = f^{(n)}(a).
\]
The Taylor series for $f$ at $a$ (or “centered at $a$”) is the formal infinite series
\[
\sum_{n=0}^{\infty} c_n(x - a)^n = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots + c_nx^n + \cdots
\]
(for now, just think of it as an infinite degree polynomial) such that when you formally differentiate it term by term, its $n$-th derivative is $f^{(n)}(a)$ for all nonnegative integers $n$.

1. Check that the third and fourth degree Taylor polynomials for $f(x) = e^x$ centered at 0 are

   \[ p_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \]

   and

   \[ p_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4. \]

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2. Set $f(x) = x^2 - 4x + 7$. Use the definition to find the Taylor polynomials of $f$ centered at 0, of degrees 0, 1, 2, 3, and 4. Call them $p_0(x), p_1(x), \ldots$. Check that they are right. Then say what the Taylor polynomials of $f$ at 0 of degree $n$ are for $n \geq 5$.

3. Set $f(x) = x^2 - 4x + 7$. (This is the same function as in the previous problem.) Use the definition to find the Taylor polynomials of $f$ centered at 1, of degrees 0, 1, 2, 3, and 4. Call them $q_0(x), q_1(x), \ldots$. Check that they are right. Then say what the Taylor polynomials of $f$ at 1 of degree $n$ are for $n \geq 5$.

Check that $q_1(x) \neq p_1(x)$, but that $q_2(x) = p_2(x)$. 
4. Set $f(x) = \frac{1}{1-x}$. Use the definition to find the Taylor polynomials of $f$ centered at 0, of degrees 0, 1, 2, 3, and 4.
5. Check that the third and fourth degree Taylor polynomials for \( f(x) = e^x \) centered at 1 are

\[
p_3(x) = e + e(x - 1) + \frac{e}{2}(x - 1)^2 + \frac{e}{6}(x - 1)^3
\]

and

\[
p_4(x) = e + e(x - 1) + \frac{e}{2}(x - 1)^2 + \frac{e}{6}(x - 1)^3 + \frac{e}{24}(x - 1)^4.
\]

Bonus problem. Set \( f(x) = \frac{1}{1-x} \). Find the Taylor series of \( f \) centered at 0.