WORKSHEET: TAYLOR POLYNOMIALS, PART 2

Recall the Taylor series for $\frac{1}{1-x}$ centered at 0:
$$1 + x + x^2 + x^3 + x^4 + x^5 + \cdots,$$
and the Taylor series for $e^x$ centered at 0:
$$1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots.$$

We will see later that the first of these converges to $\frac{1}{1-x}$ whenever $-1 < x < 1$, and the second of these converges to $e^x$ for all real $x$.

General principle (which we will see more of later): If a power series $c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots + c_nx^n + \cdots$ converges to $g(x)$ for $x$ near $a$, then it must be the Taylor series for $g(x)$ centered at 0.

1. We want to find the Taylor polynomial of degree 5 for $f(x) = x^3e^x$ centered at 0. Try computing several derivatives of $f$ at 0, say $f'(0)$ and $f''(0)$. Do you want to continue? Use a different method to find the Taylor polynomial of degree 5 for $f(x) = x^3e^x$ centered at 0.

Solution: We compute derivatives with the product rule:

$$f(0) = 0.$$

$$f'(x) = 3x^2e^x + x^3e^x = (3x^2 + x^3)e^x,$$

so $f'(0) = 0$.

$$f''(x) = (6x + 3x^2)e^x + (3x^2 + x^3)e^x = (6x + 6x^2 + x^3)e^x,$$

so $f''(0) = 0$.

If you do one more:

$$f'''(x) = (6 + 12x + 3x^2)e^x + (6x + 6x^2 + x^3)e^x = (6 + 18x + 9x^2 + x^3)e^x,$$

so $f'''(0) = 6$.
Better method: Taylor series for $x^3e^x$ centered at 0 is
\[
x^3 \left( 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots \right)
= x^3 + x^4 + \frac{1}{2}x^5 + \frac{1}{6}x^6 + \frac{1}{24}x^7 + \frac{1}{120}x^8 + \cdots,
\]
so the Taylor polynomial of degree 5 for $f(x) = x^3e^x$ centered at 0 is
\[
p_5(x) = x^3 + x^4 + \frac{1}{2}x^5.
\]

2. We want to find the Taylor polynomial of degree 8 for $f(x) = \frac{1}{1 + x^2}$ centered at 0. The derivatives of the function get messy. Find an easier way to do this. Use your answer to find $f^{(7)}(0)$ and $f^{(8)}(0)$ without finding $f'(x)$, $f''(x)$, \ldots, $f^{(8)}(x)$.

Solution: Start with
\[
\frac{1}{1 - x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots,
\]
valid for $-1 < x < 1$. Now if $-1 < x < 1$, then also $-1 < -x^2 < 1$, so
\[
\frac{1}{1 + x^2} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + (-x^2)^4 + (-x^2)^5 + \cdots
= 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots,
\]
valid for $-1 < x < 1$. So this expression must be the Taylor series for $f(x)$ centered at 0. Therefore the Taylor polynomial of degree 8 for $f(x) = \frac{1}{1 + x^2}$ centered at 0 is
\[
p_8(x) = 1 - x^2 + x^4 - x^6 + x^8.
\]
You know the coefficient of $x^7$ is supposed to be $\frac{f^{(7)}(0)}{7!}$, so it follows that $f^{(7)}(0) = 0$. You know the coefficient of $x^8$ is supposed to be $\frac{f^{(8)}(0)}{8!}$, so it follows that $f^{(8)}(0) = 8!$.

3. We want to find the Taylor polynomial of degree 4 for $f(x) = \frac{e^x}{1 - x}$ centered at 0. The derivatives of $f$ get messy quickly, so use an easier method. Don’t simplify the coefficients.
Solution: We multiply the two series at the beginning, keeping only the terms of degree at most 4. We get

\[
1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \frac{1}{5!} x^5 + \cdots
+ x \left( 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \frac{1}{5!} x^5 + \cdots \right)
+ x^2 \left( 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \frac{1}{5!} x^5 + \cdots \right)
+ x^3 \left( 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \frac{1}{5!} x^5 + \cdots \right)
+ x^4 \left( 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \frac{1}{5!} x^5 + \cdots \right)
+ x^5 \left( 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \frac{1}{5!} x^5 + \cdots \right)
+ \cdots.
\]

From the first line we keep the terms

\[
1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4.
\]

From the second line we keep the terms in

\[
x \left( 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 \right).
\]

From the third line we keep the terms in

\[
x^2 \left( 1 + x + \frac{1}{2!} x^2 \right).
\]

From the fourth line we keep the terms in

\[
x^3 (1 + x).
\]

From the fifth line we keep only the term in

\[
x^4 \cdot 1.
\]

In the sixth and all further lines, all terms have degree 5 or more, so we don’t keep any of them. So the Taylor polynomial of degree 4 centered at 0
is
\[ p_4(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + x \left( 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 \right) + x^2 \left( 1 + x + \frac{1}{2!}x^2 \right) + x^3 (1 + x) + x^4 \cdot 1 \]
\[ = 1 + 2x + \left( \frac{1}{2} + 1 + 1 \right)x^2 + \left( \frac{1}{6} + \frac{1}{2} + 1 + 1 \right)x^3 \]
\[ + \left( \frac{1}{24} + \frac{1}{6} + \frac{1}{2} + 1 + 1 \right)x^4. \]

4. Set \( f(x) = e^x \). Find \( f^{(14)}(0) \) and \( f^{(19)}(0) \).

**Solution:** We aren’t going to find \( f^{(14)}(x) \) and \( f^{(19)}(x) \). To see why, here is what my computer algebra system gave for \( f^{(19)}(x) \):
\[
\begin{align*}
4257578514309120000 \cdot e^x &+ 350036817553924300000 \cdot x^9 e^x \\
+ 4085681395917632686080000 \cdot x^{16} e^x &+ 71385287640348701859840000 \cdot x^{23} e^x \\
+ 381571584013974517670707200 \cdot x^{30} e^x &+ 863237730417502353590476800 \cdot x^{37} e^x \\
+ 989205399855972119390169600 \cdot x^{44} e^x &+ 640696151589831997937337600 \cdot x^{51} e^x \\
+ 2516794557083870556952000 \cdot x^{58} e^x &+ 62769614229090664246391040 \cdot x^{65} e^x \\
+ 102223826851036169348160 \cdot x^{72} e^x &+ 1101822889740173260571520 \cdot x^{79} e^x \\
+ 7852753757605720345920 \cdot x^{86} e^x &+ 3632465656635212023200 \cdot x^{93} e^x \\
+ 104143563240764855346 \cdot x^{100} e^x &+ 1670752351456120674 \cdot x^{107} e^x \\
+ 11398895185373143 \cdot x^{114} e^x &\text{.}
\end{align*}
\]

Instead, since
\[
e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots,
\]
valid for all real \( x \), we have
\[
e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots,
\]
valid for all real \( x \). The coefficient of \( x^n \) is \( \frac{f^{(n)}(0)}{n!} \). Since the coefficient of \( x^{19} \) is zero, clearly \( f^{(19)}(0) = 0 \). The coefficient of \( x^{14} \) is \( \frac{1}{2} \), so
\[
\frac{1}{2} = \frac{f^{(14)}(0)}{14!},
\]
whence \( f^{(14)}(0) = 14! / 2 \).