1. (3 points/part.) For each of the functions below, find the requested Taylor polynomial centered at the given point.

   (a) The 7th degree Taylor polynomial for \( f(x) = \sin(x) \), centered at \( x = 0 \).

   (b) The 6th degree Taylor polynomial for \( f(x) = \cos(x) \), centered at \( x = 0 \).

   (c) The 3rd degree Taylor polynomial for \( f(x) = \frac{1}{1 + 3x} \), centered at \( x = 0 \).

   (d) The 6th degree Taylor polynomial for \( f(x) = \ln(1 + x) \), centered at \( x = 0 \).

   (e) The 3rd degree Taylor polynomial for \( f(x) = \frac{1}{2 + x} \), centered at \( x = 0 \).

   (f) The 3rd degree Taylor polynomial for \( f(x) = \frac{1}{1 - x} \), centered at \( x = 3 \) (not \( x = 0 \)).

   (g) The 3rd degree Taylor polynomial for \( f(x) = \sqrt{x} \), centered at \( x = 4 \) (not \( x = 0 \)).

2. (4 points/part.) For each of the functions below, find the requested Taylor polynomial centered at the given point.

   (a) The 3rd degree Taylor polynomial for \( f(x) = \sin^2(x) \), centered at \( x = 0 \).

   (b) The 12th degree Taylor polynomial for \( f(x) = \sin(x^4) \), centered at \( x = 0 \).

   (c) The 4th degree Taylor polynomial for \( f(x) = xe^{3x} \), centered at \( x = 0 \).

   (d) The 3rd degree Taylor polynomial for \( f(x) = \frac{x}{e^x} \), centered at \( x = 0 \).

   (e) The 3rd degree Taylor polynomial for \( f(x) = \frac{e^x}{\cos(x)} \), centered at \( x = 0 \).

   (f) The 3rd degree Taylor polynomial for \( f(x) = \frac{1}{1 - 2x} \), centered at \( x = 0 \).

3. (9 points.) Starting with \( 12^3 = 1728 \), approximate \( \sqrt[3]{1729} \) using a degree two Taylor polynomial for \( f(x) = \sqrt[3]{x} \) centered at \( x = 1728 \). How many decimal places will stay the same if you use a degree three Taylor polynomial instead?

   (Use a calculator and give decimal approximations.)

4. (8 points.) Suppose you want to approximate \( \sin(0.5) \) using a Taylor polynomial centered at \( x = 0 \). What is the smallest degree Taylor polynomial you can use that will give the correct answer up to eight decimal places?

   (Use a calculator and compare with what it says \( \sin(0.5) \) is. We will learn better methods later in the course.)
5. (5 points/part.) The function \( x \cos(x^3) \) has an antiderivative (you know this from Math 252), but doesn’t have an antiderivative which is an elementary function (the kind of function you have been working with in math courses so far). So you can’t find \( \int_0^1 x \cos(x^3) \, dx \) in the way you learned in Math 252. Instead, use Taylor polynomials for \( \cos(x) \) to approximate the above integral without using a calculator or numerical integration software.

(Computer calculations show that \( \int_0^1 x \cos(x^3) \, dx \approx 0.440408 \).)

(a) Use the degree 2 Taylor polynomial for \( \cos(x) \) centered at 0.

(b) Use the degree 4 Taylor polynomial for \( \cos(x) \) centered at 0.

6. (4 points/part.) Define

\[ h(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0. \end{cases} \]

(The letter “\( h \)” stands for “horrible”, as you will see in this problem.)

We are going to investigate the Taylor polynomials for \( h(x) \) centered at 0. We will need to find the derivatives of \( h \).

(a) Find \( \lim_{x \to \infty} e^{-x^2} \) and \( \lim_{x \to -\infty} e^{-x^2} \).

(b) Use the limits in part (a) to show that

\[ \lim_{x \to 0^+} e^{-1/x^2} = 0 \quad \text{and} \quad \lim_{x \to 0^-} e^{-1/x^2} = 0. \]

Why does this show that \( h \) is continuous at 0?

(c) Find \( \lim_{x \to \infty} xe^{-x^2} \) and \( \lim_{x \to -\infty} xe^{-x^2} \). (One way to do this is to use L’Hopital’s Rule.) Then find:

\[ \lim_{x \to \infty} x^2e^{-x^2}, \quad \lim_{x \to \infty} x^3e^{-x^2}, \quad \text{and} \quad \lim_{x \to \infty} x^4e^{-x^2}, \]

and the corresponding limits with \( x \to -\infty \). For positive integers \( n \), what do you think \( \lim_{x \to \infty} x^n e^{-x^2} \) and \( \lim_{x \to -\infty} x^n e^{-x^2} \) are? (No work required for this last statement.)

(d) Use the first two limits in part (c) to show that

\[ \lim_{x \to 0^+} \frac{e^{-1/x^2}}{x} = 0 \quad \text{and} \quad \lim_{x \to 0^-} \frac{e^{-1/x^2}}{x} = 0. \]

Why does this show that \( h'(0) = 0 \)? (You will need to use the definition of the derivative.)

(e) Find \( h'(x) \) for \( x \neq 0 \).

(f) Use suitable limits from part (c) to show that \( h''(0) = 0 \). (Like in part (d), you will have to do this from the definition of the derivative.)

(g) Similar reasoning (getting more complicated with each derivative) shows that \( h^{(n)}(0) = 0 \) for all positive integers \( n \). Given this, what is the \( n \)-th degree Taylor polynomial for \( h \) centered at 0? How well do these Taylor polynomials approximate the function \( h \)?