1. Find the exact value of \( \lim_{n \to \infty} \frac{(n + 1)!}{3n \cdot n!} \).

**Solution:** For every positive integer \( n \), we have, cancelling at the second step,

\[
\frac{(n + 1)!}{3n \cdot n!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots n \cdot (n + 1)}{3 \cdot n \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdots n} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots n \cdot (n + 1)}{3 \cdot n} = \frac{n + 1}{3n} = \frac{1 + \frac{1}{n}}{3}.
\]

So

\[
\lim_{n \to \infty} \frac{(n + 1)!}{3n \cdot n!} = \lim_{n \to \infty} \frac{1 + \frac{1}{n}}{3} = \frac{1}{3}.
\]

2. Set \( a_n = 1 - \frac{1}{n} \) for positive integers \( n \). Which of these statements is correct? (First, explicitly calculate the first few terms.)

(1) \( \{a_n\}_{n=1}^{\infty} \) is bounded.
(2) \( \{a_n\}_{n=1}^{\infty} \) is strictly increasing. (The book has “increasing”.)
(3) \( \{a_n\}_{n=1}^{\infty} \) is strictly decreasing. (The book has “decreasing”.)
(4) \( \{a_n\}_{n=1}^{\infty} \) is strictly monotonic. (The book has “monotonic”.)

**Solution:**

(1): Yes: \( 0 \leq a_n \leq 1 \) for all positive integers \( n \).
(2): Yes: for all positive integers \( n \), \( n + 1 > n > 0 \), so \( \frac{1}{n+1} < \frac{1}{n} \), so

\[
a_{n+1} = 1 - \frac{1}{n+1} > 1 - \frac{1}{n} = a_n.
\]

(3): No: \( a_1 = 0 \) and \( a_2 = \frac{1}{2} \), so it isn’t true that \( a_2 < a_1 \).
(4): Yes, by (2).

3. Set \( b_n = n + (-1)^n \cdot n + 2 \) for positive integers \( n \). Which of these statements is correct? (First, explicitly calculate the first few terms.)

(1) \( \{b_n\}_{n=1}^{\infty} \) is bounded.
(2) \( \{b_n\}_{n=1}^{\infty} \) is strictly increasing.
(3) \( \{b_n\}_{n=1}^{\infty} \) is strictly decreasing.
(4) \( \{b_n\}_{n=1}^{\infty} \) is strictly monotonic.

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Solution: The sequence is 
 \[(2, 6, 2, 10, 2, 14, 2, \ldots).\]

(1): No: if \(n\) is even then \(b_n = 2n + 2\), and \(\lim_{n \to \infty} (2n + 2) = \infty\). (However, \((b_n)_{n=1}^{\infty}\) is bounded below, because \(b_n > 0\) for all positive integers \(n\).)

(2): No: \(b_2 \leq b_3\).

(3): No: \(b_1 \geq b_2\).

(4): No, by (2) and (3).

4. Define a sequence \((a_n)_{n=1}^{\infty}\) by \(a_n = \sqrt{n}\) for \(n = 1, 2, \ldots\). You want to prove that \(\lim_{n \to \infty} a_n = 2\). So, for example, for \(M = 137\) you should be able to find some integer \(N > 0\) such that for all \(n > N\) we have \(a_n > M\). Find such a value of \(N\), and show that it works. (You need not find the best value of \(N\).)

Solution: I am going to choose \(N = 40,000\). (This is not the best choice.) Consider an arbitrary integer \(n\) such that \(n > 40,000\). Then \(a_n = \sqrt{n} > \sqrt{40,000} = 200\). In particular, \(a_n > 137\). So \(N = 40,000\) works.

(For reference, the smallest choice which works is \(N = 137^2 = 18,769\).)

5. Define a sequence \((x_n)_{n=1}^{\infty}\) by
\[x_n = \frac{2n + 1}{n}\]
for \(n = 1, 2, \ldots\). You want to prove that \(\lim_{n \to \infty} x_n = 2\). So, for example, for \(\varepsilon = 0.2\) you should be able to find some integer \(N > 0\) such that for all \(n > N\) we have \(|x_n - 2| < \varepsilon\). Find such a value of \(N\), and show that it works. (You need not find the best value of \(N\).)

Solution: Rewrite
\[x_n = 2 + \frac{1}{n}\]
for \(n = 1, 2, \ldots\). So
\[|x_n - 2| = \left|\frac{1}{n}\right| = \frac{1}{n}.\]
I am going to choose \( N = 10 \). (This is not the best choice.) Consider an arbitrary integer \( n \) such that \( n > 10 \). Then
\[
|x_n - 2| = \frac{1}{n} < \frac{1}{10} = 0.1 < 0.2.
\]
So \( N = 30 \) works.

(For reference, the smallest choice which works is \( N = 5 \).)

6. Define a sequence \( (c_n)_{n=1}^{\infty} \) by
\[
c_n = \frac{n^2 - 10}{n^2}
\]
for \( n = 1, 2, \ldots \). You want to prove that \( \lim_{n \to \infty} c_n = 1 \). So, for example, for \( \varepsilon = 0.1 \) you should be able to find some integer \( N > 0 \) such that for all \( n > N \) we have \( |c_n - 1| < \varepsilon \). Find such a value of \( N \), and show that it works. (You need not find the best value of \( N \).)

Rewrite
\[
c_n = 1 - \frac{10}{n^2}
\]
for \( n = 1, 2, \ldots \). So
\[
|c_n - 1| = \left| -\frac{10}{n^2} \right| = \frac{10}{n^2}.
\]
I am going to choose \( N = 30 \). (This is not the best choice.) Consider an arbitrary integer \( n \) such that \( n > 30 \). Then
\[
|c_n - 1| = \frac{10}{n^2} < \frac{10}{30^2} = \frac{10}{900} = \frac{10}{100} = 0.1.
\]
So \( N = 30 \) works.

(For reference, the smallest choice which works is \( N = 10 \).)