1. (Section 8.2 problem 12 in the textbook.) Determine whether the series is convergent or divergent. If it is convergent, find the sum.

\[ 4 + 3 + \frac{9}{4} + \frac{27}{16} + \cdots. \]

(Hint: It is a geometric series.)

2. (Section 8.2 problem 14 in the textbook.) Determine whether the series is convergent or divergent. If it is convergent, find the sum.

\[ 1 + 0.4 + 0.16 + 0.064 + \cdots. \]

(Hint: It is a geometric series.)

3. (Section 8.2 problem 16 in the textbook.) Determine whether the series is convergent or divergent. If it is convergent, find the sum.

\[ \sum_{n=1}^{\infty} \frac{10^n}{(-9)^{n-1}}. \]

(Hint: It is a geometric series.)

4. (Section 8.2 problem 18 in the textbook.) Determine whether the series is convergent or divergent. If it is convergent, find the sum.

\[ \sum_{n=0}^{\infty} \frac{1}{(\sqrt{2})^n}. \]

(Hint: It is a geometric series.)

5. (Section 8.2 problem 20 in the textbook.) Determine whether the series is convergent or divergent. If it is convergent, find the sum.

\[ \sum_{k=1}^{\infty} \frac{k(k + 2)}{(k + 3)^2}. \]

6. (Section 8.2 problem 22 in the textbook.) Determine whether the series is convergent or divergent. If it is convergent, find the sum.

\[ \sum_{n=1}^{\infty} \cos \left( \frac{1}{n} \right). \]

7. (Section 8.2 problem 24 in the textbook.) Determine whether the series is convergent or divergent. If it is convergent, find the sum.

\[ \sum_{n=1}^{\infty} \frac{1 + 3^n}{2^n}. \]

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8. (Section 8.2 problem 26 in the textbook.) Determine whether the series is convergent or divergent. If it is convergent, find the sum.

\[ \sum_{k=1}^{\infty} \cos(1)^k. \]

9. (Section 8.2 problem 28 in the textbook.) Determine whether the series is convergent or divergent. If it is convergent, find the sum.

\[ \sum_{n=1}^{\infty} \left[ (0.8)^{n-1} - (0.3)^n \right]. \]

10. (Section 8.2 problem 30 in the textbook.) Determine whether the series is convergent or divergent. If it is convergent, find the sum.

\[ \sum_{n=1}^{\infty} \left( \frac{3}{5^n} + \frac{2}{n} \right). \]

11. (Section 8.2 problem 32 in the textbook.) Determine whether the series is convergent or divergent. If it is convergent, find the sum.

\[ \sum_{n=1}^{\infty} \frac{2}{n^2 + 4n + 3}. \]

(Hint: Express the partial sums as telescoping sums.)

12. (Section 8.2 problem 33 in the textbook.) Determine whether the series is convergent or divergent. If it is convergent, find the sum.

\[ \sum_{n=1}^{\infty} \frac{3}{n(n+3)}. \]

(Hint: Express the partial sums as telescoping sums.)

13. (Section 8.2 problem 34 in the textbook.) Determine whether the series is convergent or divergent. If it is convergent, find the sum.

\[ \sum_{n=1}^{\infty} \ln \left( \frac{n}{n+1} \right). \]

(Hint: Express the partial sums as telescoping sums.)

14. (Section 8.2 problem 36 in the textbook.) Express 0.7373737373\ldots as a ratio of integers.

15: Section 8.2 problem 50 in the textbook. (Not reproduced here; see the book.)

16: Section 8.2 problem 52a in the textbook. (Not reproduced here; see the book.)

Your answer will depend on \( H \) and \( r \). Check that if \( r = \frac{1}{3} \) then the ball travels a total distance of \( 2H \).

17. (Section 8.3 problem 2 in the textbook.) Suppose that \( f \) is a strictly positive strictly decreasing function, defined on \([1, \infty)\). For positive integers \( n \), set \( a_n = f(n) \). By drawing a picture, arrange the following three quantities in increasing order:

\[ \int_{1}^{6} f(x) \, dx, \quad \sum_{k=1}^{5} a_k, \quad \sum_{k=2}^{6} a_k. \]
18. (Section 8.3 problem 6 in the textbook.) Use the Integral Test to determine whether the series \( \sum_{n=1}^{\infty} \frac{1}{n^5} \) is convergent or divergent.

19. (Section 8.3 problem 8 in the textbook.) Use the Integral Test to determine whether the series \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + 4} \) is convergent or divergent.

20. (Section 8.3 problem 12 in the textbook.) Determine whether the series \( \sum_{n=1}^{\infty} (n^{-1.4} + 3n^{-1.2}) \) is convergent or divergent.

21. (Section 8.3 problem 13 in the textbook.) Determine whether the series \( 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots \) is convergent or divergent.

22. (Section 8.3 problem 14 in the textbook.) Determine whether the series \( 1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \cdots \) is convergent or divergent.

23. (Section 8.3 problem 18 in the textbook.) Determine whether the series \( \sum_{n=1}^{\infty} \frac{1}{n^2 + 9} \) is convergent or divergent.

24. (Section 8.3 problem 23 in the textbook.) Determine whether the series \( 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots \) is convergent or divergent.

25. Define functions \( F \) and \( G \) by \( F(x) = \frac{e^x + e^{-x}}{2} \) and \( G(x) = \frac{e^x - e^{-x}}{2} \) for all real numbers \( x \). Find the degree 10 Taylor expansions of \( F(x) \) and \( G(x) \) centered at \( x = 0 \). What familiar functions do these Taylor series seem related to?

26. Approximate \( \int_{-1}^{1} \frac{\sin(x)}{x} \, dx \) using a fifth degree Taylor polynomial for \( \sin(x) \). (The actual value of the integral is about 1.89217.)

27. Define a sequence \( (a_n)_{n=1}^{\infty} \) by
\[
a_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln(n + 1)
\]
for positive integers \( n \). Show that \( (a_n)_{n=1}^{\infty} \) converges.

Caution: This problem is about the limit of a sequence. There is no series anywhere in the problem. To illustrate, the first few terms are:
\[
a_1 = 1 - \ln(2).
a_2 = 1 + \frac{1}{2} - \ln(3) = \frac{3}{2} - \ln(3),
a_3 = 1 + \frac{1}{2} + \frac{1}{3} - \ln(4) = \frac{11}{6} - \ln(4).
\]
\[a_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \ln(5) = \frac{25}{12} - \ln(5).\]
\[a_5 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \ln(6) = \frac{137}{60} - \ln(6).\]

Hint: Show that \((a_n)_{n=1}^{\infty}\) is monotonic and bounded.

Here are two pictures which show you why \((a_n)_{n=1}^{\infty}\) is monotonic, and which, after drawing some extra lines, also can be used to show why \((a_n)_{n=1}^{\infty}\) is bounded. In both pictures, the curve is the graph of \(y = \frac{1}{x}\). In the first, \(a_3\) is the sum of the areas of the three black shaded regions. In the second, \(a_6\) is the sum of the areas of the six black shaded regions.

Extra credit. As in the previous problem, define a sequence \((a_n)_{n=1}^{\infty}\) by
\[a_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln(n + 1)\]
for positive integers \(n\). Show that \(\lim_{n \to \infty} a_n > \frac{1}{2}\).

(This limit is called the Euler constant, written \(\gamma\). It is approximately
\[\gamma \approx 0.577215664901532860606512090082402431042.\]
It is not known whether \(\gamma\) is a rational number. The conventional definition is
\[\gamma = \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln(n) \right).\]
Since \(\lim_{n \to \infty}(\ln(n + 1) - \ln(n)) = 0\), this limit has the same value.)