1. Find $\sum_{n=1}^{\infty} \frac{47}{100^n}$.

*Solution:* This is a geometric series with common ratio $\frac{1}{100}$. So:

$$\sum_{n=1}^{\infty} \frac{47}{100^n} = \frac{47}{100} \sum_{m=0}^{\infty} \frac{1}{100^m} = \left( \frac{47}{100} \right) \left( \frac{1}{1 - \frac{1}{100}} \right) = \left( \frac{47}{100} \right) \left( \frac{100}{99} \right) = \frac{47}{99}.$$

2. What does Problem 1 tell you about the number $b$ whose decimal expansion is $0.47474747\ldots$? What does Problem 1 tell you about the number $c$ whose decimal expansion is $2.47474747\ldots$?

*Solution:* We have

$$b = \sum_{n=1}^{\infty} \frac{47}{100^n} = \frac{47}{99} \quad \text{and} \quad c = 2 + b = 2 + \frac{47}{99} = \frac{245}{99}.$$

3. Express the number with decimal expansion $0.182182182\ldots$ as a ratio of integers.

*Solution:* We write, using the formula for the sum of a geometric series at the third step,

$$0.182182182\ldots = \sum_{n=1}^{\infty} \frac{182}{1000^n} = \frac{182}{1000} \sum_{m=0}^{\infty} \frac{1}{1000^m} = \left( \frac{182}{1000} \right) \left( \frac{1}{1 - \frac{1}{1000}} \right) = \left( \frac{182}{1000} \right) \left( \frac{1000}{999} \right) = \frac{182}{999}.$$

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4. Use the Integral Test to determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$ converges. You will use a suitable function $f$; make sure to check that your choice of $f$ satisfies the two conditions it is supposed to.

Solution: We take $f(x) = \frac{1}{x^2 + 1}$. We need to check that $f(x) \geq 0$ beyond some point; this is clearly true for all real $x$. We need to check that $f$ is nonincreasing beyond some point; since $x^2 + 1 > 0$ and is increasing on $[0, \infty)$, clearly $\frac{1}{x^2 + 1}$ is decreasing on $[0, \infty)$.

Now

$$\int_0^{\infty} \frac{1}{x^2 + 1} \, dx = \arctan(x) \bigg|_0^{\infty} = \lim_{x \to \infty} \arctan(x) - \arctan(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$ 

So the improper integral converges. Therefore $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$ converges.

5. Use the Integral Test to determine whether the series $\sum_{n=1}^{\infty} \frac{[\ln(n)]^2}{n}$ converges. You will use a suitable function $f$; make sure to check that your choice of $f$ satisfies the two conditions it is supposed to.

Hint: One way to check that a function is decreasing is to look at its derivative.

Solution: We take $f(x) = \frac{[\ln(x)]^2}{x}$. We need to check that $f(x) \geq 0$ beyond some point; this is clearly true for all $x > 0$. We also need to check that $f$ is nonincreasing beyond some point. For this, we calculate

$$f'(x) = \frac{2 \ln(x) \cdot \frac{1}{x} x - [\ln(x)]^2}{x^2} = \frac{2 \ln(x) - [\ln(x)]^2}{x^2} = \frac{[2 - \ln(x)] \ln(x)}{x^2}.$$ 

If $x > e^2$ then $\ln(x) > 2$, so $f'(x) < 0$. So $f$ is decreasing on $(e^2, \infty)$.

Using the substitution $u = \ln(x)$, so $du = \frac{1}{x} \, dx$, we get

$$\int_1^{\infty} \frac{[\ln(x)]^2}{x} \, dx = \left( \frac{1}{3} [\ln(x)]^3 \right) \bigg|_1^{\infty}.$$ 

Since $\lim_{x \to \infty} \frac{1}{3} [\ln(x)]^3 = \infty$, the improper integral diverges. Therefore $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$ diverges.
6. Use the Integral Test to determine whether the series \( \sum_{n=1}^{\infty} \frac{1}{n^4 + 2} \) converges. You will use a suitable function \( f \); make sure to check that your choice of \( f \) satisfies the two conditions it is supposed to.

Hint: You can’t integrate the appropriate function \( f \) in a reasonable time. (It does, however, have an elementary antiderivative.) Instead, use the Comparison Test for improper integrals to determine whether \( \int_{1}^{\infty} f(x) \, dx \) is convergent.

**Solution:** We take \( f(x) = \frac{1}{x^4 + 2} \). We need to check that \( f(x) \geq 0 \) beyond some point; this is clearly true for all real \( x \). We need to check that \( f \) is nonincreasing beyond some point; since \( x^4 + 2 > 0 \) and is increasing on \([0, \infty)\), clearly \( \frac{1}{x^4 + 2} \) is decreasing on \([0, \infty)\).

Since \( \frac{1}{x^4 + 2} \) looks like \( \frac{1}{x^4} \) for large \( x \), we guess that \( \int_{1}^{\infty} \frac{1}{x^4 + 2} \, dx \) is convergent.

For all \( x \) in \([1, \infty)\), we have \( 0 < x^4 < x^4 + 2 \), so
\[
0 < \frac{1}{x^4 + 2} < \frac{1}{x^4}.
\]

Now
\[
\int_{1}^{\infty} \frac{1}{x^4} \, dx = \left[ \frac{1}{3x^3} \right]_{1}^{\infty} = \lim_{x \to \infty} \left( -\frac{1}{3x^3} \right) - \left( -\frac{1}{3} \right) = 0 + \frac{1}{3} = \frac{1}{3}.
\]
So \( \int_{1}^{\infty} \frac{1}{x^4} \, dx \) converges. (Actually, you know already that this improper integral converges.)

Therefore \( \int_{0}^{\infty} \frac{1}{x^4 + 2} \, dx \) converges by the Comparison Test for improper integrals. So \( \sum_{n=1}^{\infty} \frac{1}{n^4 + 2} \) converges.

7. We already know that \( \sum_{n=1}^{\infty} \frac{1}{n^4} \) is convergent. How can you use this fact to show that \( \sum_{n=1}^{\infty} \frac{1}{n^4 + 2} \) is convergent without using the Integral Test?

**Solution:** We have
\[
0 < \frac{1}{x^4 + 2} < \frac{1}{x^4}
\]
for \( n = 1, 2, 3, \ldots \). Let \( s_n \) be the \( n \)-th partial sum for the series \( \sum_{n=1}^{\infty} \frac{1}{n^4 + 2} \), and let \( t_n \) be the \( n \)-th partial sum for the series \( \sum_{n=1}^{\infty} \frac{1}{n^4} \). Then \( 0 \leq s_n \leq t_n \) for \( n = 1, 2, 3, \ldots \). Since \( \{t_n\}_{n=1}^{\infty} \) is increasing and convergent, this sequence is bounded. So \( \{t_n\}_{n=1}^{\infty} \) must be bounded. Since \( \{t_n\}_{n=1}^{\infty} \) is increasing, \( \{t_n\}_{n=1}^{\infty} \) converges. That is, \( \sum_{n=1}^{\infty} \frac{1}{n^4 + 2} \) is convergent.