WORKSHEET: RATIO TEST

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Recall the Ratio Test for convergence of the series \( \sum_{n=1}^{\infty} a_n \), in which all the summands are nonzero:

1. If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \) exists and is less than 1, then \( \sum_{n=1}^{\infty} a_n \) is absolutely convergent (and therefore also convergent).
2. If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \) exists and is greater than 1, or if \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty \), then \( \sum_{n=1}^{\infty} a_n \) is divergent.
3. In all other cases, that is, if \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1 \) or if this limit fails to exist in any other way than diverging to \( \infty \), the Ratio Test says nothing. Try some other test.

1. For each of the following series, what does the Ratio Test say about its convergence? If the Ratio Test doesn’t apply, can you use other methods?

   a. \( \sum_{n=1}^{\infty} \frac{(-1)^n 15^n}{n!} \)

   Solution: Trying to apply the Ratio Test, we calculate:

   \[
   \lim_{n \to \infty} \left| \frac{\left(\frac{(-1)^{n+1} 15^{n+1}}{(n+1)!}\right)}{\left(\frac{(-1)^n 15^n}{n!}\right)} \right| = \lim_{n \to \infty} \frac{15^{n+1} n!}{15^n (n+1)!} = \lim_{n \to \infty} \frac{15}{n+1} = 0.
   \]

   Since \( 0 < 1 \), the Ratio Test applies, and says the series is absolutely convergent, and therefore also convergent.

   Note: Parentheses are required in the first expression, to make the order of the three divisions unambiguous. (The textbook sometimes omits them; this is an error.)

   b. \( \sum_{n=1}^{\infty} (-1)^{n-1} n \)

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Solution: Trying to apply the Ratio Test, we calculate:

\[
\lim_{n \to \infty} \left| \frac{(-1)^n (n + 1)}{(-1)^{n-1} n} \right| = \lim_{n \to \infty} \frac{n + 1}{n} = 1.
\]

So the Ratio Test tells us nothing. However, the series doesn’t converge, since the summands don’t approach 0. (In fact, their absolute values go to infinity.)

c. \[\sum_{n=1}^{\infty} \frac{n^{16}}{6^n}\]

Solution: Trying to apply the Ratio Test, we calculate:

\[
\lim_{n \to \infty} \left| \frac{(\frac{(n+1)^{16}}{6^{n+1}})}{(\frac{n^{16}}{6^n})} \right| = \lim_{n \to \infty} \frac{(n + 1)^{16}}{6n^{16}} = \lim_{n \to \infty} \frac{1}{6} \left( \frac{n + 1}{n} \right)^{16} = \frac{1}{6}.
\]

Since \(\frac{1}{6} < 1\), the Ratio Test applies, and says the series is absolutely convergent, and therefore also convergent.

Note: Parentheses are required in the first expression, to make the order of the three divisions unambiguous. (The textbook sometimes omits them; this is an error.)

d. \[\sum_{n=1}^{\infty} \frac{n}{n^3 + 1}\]

Solution: Trying to apply the Ratio Test, we calculate:

\[
\lim_{n \to \infty} \left| \frac{\frac{n+1}{(n+1)^3+1}}{\frac{n}{n^3+1}} \right| = \lim_{n \to \infty} \frac{(n + 1)(n^3 + 1)}{n[(n + 1)^3 + 1]}.
\]

You can check that this limit is 1; details omitted. So the Ratio Test tells us nothing.

The series is convergent by the Comparison Test, since for \(n = 1, 2, 3, \ldots\) we have

\[0 < \frac{n}{n^3 + 1} < \frac{n}{n^3} = \frac{1}{n^2},\]

and \[\sum_{n=1}^{\infty} \frac{1}{n^2}\] is convergent.