Recall the Alternating Series Test. Suppose that:

1. \( b_n > 0 \) for \( n = 1, 2, 3, \ldots \)
2. \( b_1 \geq b_2 \geq b_3 \geq \cdots \) (that is, \( b_{n+1} \leq b_n \) for \( n = 1, 2, 3, \ldots \)).
3. \( \lim_{n \to \infty} b_n = 0 \).

Then the “alternating series”

\[
\sum_{n=1}^{\infty} (-1)^{n-1} = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + b_7 - b_8 + \cdots
\]

is convergent. In fact (see picture in the book), if the partial sums are \( s_1, s_2, s_3, \ldots \) and the sum of the series is \( s \), then

\[
s_2 \leq s_4 \leq s_6 \leq \cdots \leq s \leq \cdots \leq s_5 \leq s_3 \leq s_1.
\]

In particular, if \( n \geq 2 \) is an even integer, then

(1) \( s_n \leq s \leq s_{n+1} = s_n + b_{n+1} \) and \( |s - s_n| \leq b_{n+1}. \)

1. If \( n \geq 1 \) is an odd integer, what is the analog of (1)?

2. Give an upper bound on the error that results from approximating

\[
\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2 + k + 1}
\]

using the partial sum \( s_9 = \sum_{k=1}^{9} \frac{(-1)^{k-1}}{k^2 + k + 1} \). Make sure to check that the hypotheses of the Alternating Series Test are satisfied.
3. You want to approximate
\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} - \cdots
\]
to within 0.0001. Which partial sum should you use? Why? (The sum of the series is \(e^{-1}\).)

Make sure to check that the hypotheses of the estimate you use are satisfied.

4. What is the analog of (1) and Problem 1 for the series
\[
\sum_{n=1}^{\infty} (-1)^n = -b_1 + b_2 - b_3 + b_4 - b_5 + b_6 - b_7 + b_8 - \cdots?
\]
Why?
Recall the Integral Test: if $f$ is a function defined on $[0, \infty)$ which is positive and decreasing there beyond some point, then the series $\sum_{n=1}^{\infty} f(n)$ is convergent if and only if the improper integral $\int_{1}^{\infty} f(x) \, dx$ is convergent.

Moreover, in this case, we have the following error estimate for the partial sums $s_1, s_2, s_3, \ldots$ as approximations to $\sum_{n=1}^{\infty} f(n)$ (see pictures on the board):

$$\int_{n+1}^{\infty} f(x) \, dx \leq \sum_{k=1}^{n} f(k) - s_n \leq \int_{n}^{\infty} f(x) \, dx$$

5. Give an upper bound on the error that results from approximating $\sum_{k=1}^{\infty} \frac{1}{k^2 + 1}$ using the partial sum $s_6 = \sum_{k=1}^{6} \frac{1}{k^2 + 1}$. Make sure to check that the hypotheses of the Integral Test are satisfied.

6. You want to approximate $\sum_{n=1}^{\infty} \frac{1}{n^3}$ to within 0.0001. Which partial sum should you use? Why?

Make sure to check that the hypotheses of the estimate you use are satisfied.

(Bonus problem on the back.)
Bonus problem. You want to approximate $\sum_{n=0}^{\infty} \frac{\sin(n)}{2^n}$ to within 0.001. Which partial sum should you use? Why? (Different methods will be required.)