WORKSHEET: CONVERGENCE OF TAYLOR SERIES, PART 6

Names and student IDs: Solutions \[\pi\pi\pi - \pi\pi\pi\pi\pi\]

Recall that if \(f\) is \(n + 1\) times differentiable on \([a - d, a + d]\), if \(|f^{(n+1)}(x)| \leq M\) for all \(x\) in \([a - d, a + d]\), and if we denote the \(n\)-th Taylor polynomial for \(f\) centered at \(a\) by \(T_{f,a,n}(x)\) and denote the remainder \(f(x) - T_{f,a,n}(x)\) by \(R_{f,a,n}(x)\), then

\[
|R_{f,a,n}(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!}
\]

for all \(x\) in \([a - d, a + d]\).

1. Use the remainder estimate with \(f(x) = e^x\) and \(a = 0\) to find some integer \(n\) (not necessarily the smallest \(n\)) such that \(|e - T_{f,0,n}(1)| \leq 10^{-7}\).

The trick used in most problems of this type so far will no longer work. You will have to work directly with \(n!\). You will probably want a calculator to calculate some factorials.

Solution: No matter what choice of \(n\) we use, \(f^{(n+1)}(x)\) will be \(e^x\). Since \(x \mapsto e^x\) is increasing on \([0, 1]\), we can take \(M = e\), no matter what \(n\) is. Now, by (1),

\[
|R_{f,0,n}(1)| \leq \frac{e \cdot |1|^{n+1}}{(n+1)!} = \frac{e}{(n+1)!}.
\]

We want \(e/[(n + 1)!] \leq 10^{-7}\). Since \(e < 3\), this will happen if \((n + 1)! > 3 \cdot 10^7\). My computer algebra system says \(11! = 39,916,800\), so \(n = 10\) works.

Explicitly checking:

\[
|R_{f,0,10}(1)| \leq \frac{e \cdot |1|^{11}}{11!} \leq \frac{3}{11!} = \frac{3}{39,916,800} < \frac{1}{10,000,000} = 10^{-7},
\]

as required.

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2. Use the remainder estimate with \( f(x) = e^x \) and \( a = 0 \) to find some integer \( n \) (not necessarily the smallest \( n \)) such that \( |e^x - T_{f,0,n}(x)| \leq 10^{-4} \) for every \( x \) in \([0, 2]\).

You will probably want a calculator to calculate some factorials.

**Solution:** No matter what choice of \( n \) we use, \( f^{(n+1)}(x) \) will be \( e^x \). Since \( x \mapsto e^x \) is increasing on \([0, 2]\), we can take \( M = e^2 \), no matter what \( n \) is. Now, by (1), and using \(|x| \leq 2\) at the second step,

\[
|R_{f,0,n}(x)| \leq \frac{e^2 \cdot |x|^{n+1}}{(n+1)!} \leq \frac{e^2 \cdot 2^{n+1}}{(n+1)!}.
\]

We want \( 2^{n+1}e^2/((n+1)!) \leq 10^{-4} \). Since \( e < 3 \), this will happen if \((n+1)!/2^{n+1} > 9 \cdot 10^4 \). My computer algebra system says 12!/212 = 116,943.75. (This is an exact value; it is 467,775/4.) So \( n = 11 \) works.

Explicitly checking: for every \( x \) in \([0, 2]\),

\[
|R_{f,0,11}(x)| \leq \frac{e^2 \cdot |x|^{12}}{12!} \leq \frac{9 \cdot 2^{12}}{12!} = \frac{4}{51,975} < 10^{-4},
\]

as required.

3. Find the Taylor polynomial \( T_{s,4,2}(x) \) for the function \( s(x) = 1/\sqrt{x} \) (with center 4 and degree 2).

**Solution:** Calculate the first three derivatives and evaluate at 4:

\[
s(x) = x^{-1/2}, \quad s'(x) = \left( -\frac{1}{2} \right) x^{-3/2},
\]

and

\[
s''(x) = \left( -\frac{3}{2} \right) \left( -\frac{1}{2} \right) x^{-5/2},
\]

so

\[
s(4) = \frac{1}{2} \quad s'(4) = \left( -\frac{1}{2} \right) \left( \frac{1}{8} \right) = -\frac{1}{16},
\]

and

\[
s''(4) = \left( \frac{3}{4} \right) \left( \frac{1}{32} \right) = \frac{3}{128}.
\]

So the Taylor polynomial is

\[
T_{s,4,2}(x) = s(4) + s'(4)(x - 4) + \frac{s''(4)}{2!}(x - 4)^2
\]

\[
= \frac{1}{2} - \frac{1}{16}(x - 4) + \frac{3}{256}(x - 4)^2.
\]
One can also get $T_{s,4,2}(x)$ from the binomial expansion: expand
\[ x^{-1/2} = (4 + (x - 4))^{-1/2}. \]

4. Use the remainder estimate to estimate the accuracy of the approximation in Problem 3 for $x$ in $[4, 4.2]$.

**Solution:** We will need
\[ s'''(x) = \left(-\frac{5}{2}\right) \left(-\frac{3}{2}\right) \left(-\frac{1}{2}\right) x^{-7/2} = -\left(\frac{15}{8}\right) x^{-7/2}. \]
The absolute value of this function is decreasing on $[4, 4.2]$, so the maximum absolute value on $[4, 4.2]$ occurs at $x = 4$, and is
\[ M = \left(\frac{15}{8}\right) 4^{-7/2} = \left(\frac{15}{8}\right) \left(\frac{1}{2^7}\right) = \frac{15}{2^{11}}. \]
Therefore, for $x$ in $[4, 4.2]$, we have
\[ |R_{s,4,2}(x)| \leq \frac{M|x - 4|^3}{3!} \leq \left(\frac{15}{2^{11}}\right) \left(\frac{1}{3!}\right) (0.2)^3 \]
\[ = \left(\frac{5}{2^{11}}\right) \left(\frac{1}{5^3}\right) = \frac{1}{2^{11} \cdot 5^2} = \frac{1}{51,200}. \]

5. (If you have time; otherwise, look at the solution afterwards.)
Suppose you know ahead of time that you have a function $f$ which is given by a convergent power series $f(x) = \sum_{n=0}^{\infty} c_n x^n$ for some (unknown) coefficients $c_0, c_1, c_2, \ldots$, with convergence on an open interval $(-r, r)$. Suppose further that you know that $f'(x) = 2f(x)$ for all real $x$, and that $f(0) = 1$.
Determine $c_0$. (Find $f(0)$ from the series, find $f(0)$ from the initial condition of the differential equation, and compare them.)
Can you find $c_1$? How about $c_2$?

**Solution:** Using the series, $f(0) = c_0$. Using the initial condition of the differential equation, $f(0) = 1$. Therefore $c_0 = f(0) = 1$.
Next, using the series, $f'(0) = c_1$. Using the differential equation, $f'(0) = 2f(0) = 2c_0 = 2$. Putting these together gives $c_1 = f'(0) = 2$. 

\[ |R_{s,4,2}(x)| \leq \frac{M|x - 4|^3}{3!} \leq \left(\frac{15}{2^{11}}\right) \left(\frac{1}{3!}\right) (0.2)^3 \]
\[ = \left(\frac{5}{2^{11}}\right) \left(\frac{1}{5^3}\right) = \frac{1}{2^{11} \cdot 5^2} = \frac{1}{51,200}. \]
To find \( c_2 \), it is easiest to do the following. The series for \( f'(x) \) is obtained by summand by summand differentiation, and is
\[
f'(x) = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \cdots.
\]
The series for \( 2f(x) \) is
\[
2f(x) = 2c_0 + 2c_1x + 2c_2x^2 + 2c_3x^3 + \cdots.
\]
Both series converge to the stated functions at least on \((-r,r)\). Since power series are unique, and the differential equation says \( f'(x) = 2f(x) \), the coefficients must agree, that is,
\[
2c_0 = c_1, \quad 2c_1 = 2c_2, \quad 2c_2 = 3c_3, \quad 2c_3 = 4c_4, \quad \ldots.
\]
In particular, \( 2c_2 = 2c_1 \), so \( c_2 = c_1 = 2 \).