Suppose you know ahead of time that you have a function \( f \) which is given by a convergent power series \( f(x) = \sum_{n=0}^{\infty} c_n x^n \) for some (unknown) coefficients \( c_0, c_1, c_2, \ldots \), with convergence on an open interval \((-r, r)\). Suppose further that you know that \( f'(x) = 13f(x) \) for all real \( x \), and that \( f(0) = 1 \).

1. In the situation above, determine \( c_0 \).
   Hint: Find \( f(0) \) from the series, find \( f(0) \) from the initial condition of the differential equation, and compare them.

2. In the situation above, determine \( c_1 \).
   Hint: Find \( f'(0) \) from the series, find \( f'(0) \) from the differential equation, and compare them. You will need the value of \( c_0 \).

3. In the situation above, determine \( c_1 \) in a slightly different way, by comparing the series for \( 13f(x) \) and \( f'(x) \).

4. In the situation above, determine \( c_2 \).
5. In the situation above, determine \(c_3, c_4, c_5, \ldots\). Do you see a pattern? What function has this Taylor series?

6. Solve the differential equation \(f'(x) = 13f(x)\) with initial condition \(f(0) = 1\) using the methods of Math 252, and see whether the result agrees with what you got above.
Now suppose you know ahead of time that you have a function $g$ which is given by a convergent power series $g(x) = \sum_{n=0}^{\infty} c_n x^n$ for some (unknown) coefficients $c_0, c_1, c_2, \ldots$, with convergence on an open interval $(-r, r)$. Suppose further that you know that $g'(x) = \frac{1}{2} x + g(x)$ for all real $x$, and that $g(0) = \frac{1}{2}$.

(This equation can’t be solved by the methods of Math 252.)

7. In the situation above, find $c_0$.

8. By comparing the series for $x + g(x)$ and $g'(x)$, determine the coefficients $c_1, c_2, c_3, c_4, c_5, \ldots$. Do you recognize a pattern? What is the function whose Taylor series you have found?

9. Show directly that the function you got in Problem 7 actually satisfies the differential equation and initial condition given above.