1. Find the terms through degree 4 of the general power series solution to the equation \( y'(x) = (x + 2)y(x) + 1 \) centered at 0.

Solution: Assume \( y(x) \) has the form \( y(x) = a_0 + a_1 x + a_2 x^2 + \cdots \). Then
\[
y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots.
\]
Substituting in \( y'(x) = (x + 2)y(x) + 1 \) gives
\[
a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = x(a_0 x + a_1 x^2 + a_2 x^3 + \cdots) + 2(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots) + 1
\]
\[
= (1 + 2a_0) + (a_0 + 2a_1)x + (a_1 + 2a_2)x^2 + (a_2 + 2a_3)x^3 + (a_3 + 2a_4)x^4 + \cdots.
\]
So we get
\[
a_1 = 1 + 2a_0,
\]
\[
a_2 = \frac{1}{2}(a_0 + 2a_1) = \frac{1}{2}(a_0 + 2(1 + 2a_0)) = \frac{1}{2}(5a_0 + 2),
\]
\[
a_3 = \frac{1}{3}(a_1 + 2a_2) = \frac{1}{3}(1 + 2a_0 + (5a_0 + 2)) = \frac{1}{3}(7a_0 + 3),
\]
and
\[
a_4 = \frac{1}{4}(a_2 + 2a_3) = \frac{1}{4}\left(\frac{1}{2}(5a_0 + 2) + \frac{2}{3}(7a_0 + 3)\right) = \frac{1}{24}(43a_0 + 18).
\]
Therefore
\[
y(x) = a_0 + (1 + 2a_0)x + \frac{1}{2}(5a_0 + 2)x^2 + \frac{1}{3}(7a_0 + 3)x^3 + \frac{1}{24}(43a_0 + 18)x^4 + \cdots.
\]

2. For \( n = 1, 2, 3, 4, \ldots \), give an equation for \( a_{n+1} \) in terms of \( a_0, a_1, \ldots, a_n \).
(This is a “recursive equation” for the coefficients. In this case, you only need \( a_n \) and \( a_{n-1} \) on the right.)

Date: 30 May 2018.
Solution: For \( n = 1, 2, 3, 4, \ldots \), the coefficient of \( x^n \) in \( y'(x) \) is \( (n+1)a_{n+1} \) and the coefficient of \( x^n \) in \( (x+2)y(x) + 1 \) is \( a_{n-1} + 2a_n \). These must be equal, so the equation is

\[
a_{n+1} = \frac{1}{n+1}(a_{n-1} + 2a_n).
\]

Note: \( n = 0 \) is a special case.
Note: You used the cases \( n = 1, 2, 3 \) in Problem 1.

3. Find the terms through degree 4 of the general power series solution to the equation \( y'(x) = xy(x) + 1 \) centered at 2 (note: not centered at 0).

Hint: You will need to figure out a way to multiply a power series in \( x - 2 \) by \( x \) and get a power series in \( x - 2 \). In more detail, suppose

\[
y(x) = a_0 + a_1(x - 2) + a_2(x - 2)^2 + a_3(x - 2)^3 + \cdots.
\]

Then it is true that

\[
xy(x) + 1 = 1 + a_0x + a_1x(x - 2) + a_2x(x - 2)^2 + a_3x(x - 2)^3 + \cdots.
\]

Unfortunately, this series is not a series in powers of \( x - 2 \), because, for example, \( a_2x(x - 2)^2 \) isn’t of the form constant times \((x - 2)^n\) for any \( n \). Further work is needed.

You have already dealt with a related situation, for example in a problem asking you to find a power series centered at 0 for the function \( f(x) = (x + 2) \sin(x) \), starting from the power series centered at 0 for \( \sin(x) \).

Solution: In the equation, replace \( x \) by its Taylor series centered at 2, which is \((x - 2) + 2\). This gives the differential equation

\[
y'(x) = [(x - 2) + 2]y(x) + 1.
\]

Now assume \( y(x) \) has the form

\[
y(x) = a_0 + a_1(x - 2) + a_2(x - 2)^2 + \cdots.
\]

Then

\[
y'(x) = a_1 + 2a_2(x - 2) + 3a_3(x - 2)^2 + 4a_4(x - 2)^3 + \cdots.
\]

Substituting in \( y'(x) = [(x - 2) + 2]y(x) + 1 \) gives

\[
a_1 + 2a_2(x - 2) + 3a_3(x - 2)^2 + 4a_4(x - 2)^3 + \cdots
\]

\[
= (x - 2)(a_0(x - 2) + a_1(x - 2)^2 + a_2(x - 2)^3 + a_3(x - 2)^4 + \cdots)
\]

\[
+ 2(a_0 + a_1(x - 2) + a_2(x - 2)^2 + a_3(x - 2)^3 + a_4(x - 2)^4 + \cdots) + 1
\]

\[
= (1 + 2a_0) + (a_0 + 2a_1)(x - 2) + (a_1 + 2a_2)(x - 2)^2
\]

\[
+ (a_2 + 2a_3)(x - 2)^3 + (a_3 + 2a_4)(x - 2)^4 + \cdots.
\]
So we get (just as in Problem 1)

\[ a_1 = 1 + 2a_0, \quad a_2 = \frac{1}{2}(a_0 + 2a_1) = \frac{1}{2}(a_0 + 2(1 + 2a_0)) = \frac{1}{2}(5a_0 + 2), \]
\[ a_3 = \frac{1}{3}(a_1 + 2a_2) = \frac{1}{3}(1 + 2a_0 + (5a_0 + 2)) = \frac{1}{3}(7a_0 + 3), \]
and
\[ a_4 = \frac{1}{4}(a_2 + 2a_3) = \frac{1}{4} \left( \frac{1}{2}(5a_0 + 2) + \frac{2}{3}(7a_0 + 3) \right) = \frac{1}{24}(43a_0 + 18). \]

Therefore

\[ y(x) = a_0 + (1 + 2a_0)(x - 2) + \frac{1}{2}(5a_0 + 2)(x - 2)^2 \]
\[ + \frac{1}{3}(7a_0 + 3)(x - 2)^3 + \frac{1}{24}(43a_0 + 18)(x - 2)^4 + \cdots. \]

Alternate solution (sketch): Instead of rewriting the differential equation, proceed in the usual way to get

\[ a_1 + 2a_2(x - 2) + 3a_3(x - 2)^2 + 4a_4(x - 2)^3 + \cdots \]
\[ = x(a_0(x - 2) + a_1(x - 2)^2 + a_2(x - 2)^3 + a_3(x - 2)^4 + \cdots) + 1. \]

Now expand \( x = (x - 2) + 2 \) and distribute, getting

\[ x(a_0(x - 2) + a_1(x - 2)^2 + a_2(x - 2)^3 + a_3(x - 2)^4 + \cdots) + 1 \]
\[ = (1 + 2a_0) + (a_0 + 2a_1)(x - 2) + (a_1 + 2a_2)(x - 2)^2 \]
\[ + (a_2 + 2a_3)(x - 2)^3 + (a_3 + 2a_4)(x - 2)^4 + \cdots. \]

Proceed as in the first solution.

Second alternate solution (sketch): Our equation says \( y'(t) = ty(t) + 1. \) Putting \( t = x + 2, \) we get

\[ (1) \quad y'(x + 2) = (x + 2)y(x + 2) + 1. \]

Define a function \( z \) by \( z(x) = y(x + 2). \) Using the Chain Rule, we get
\[ z'(x) = y'(x + 2) \cdot 1 = y'(x + 2). \] So
\[ z'(x) = (x + 2)z(x) + 1. \]

This is the equation from Problem 1 (except with a different name for the function), where we got

\[ z(x) = a_0 + (1 + 2a_0)x + \frac{1}{2}(5a_0 + 2)x^2 \]
\[ + \frac{1}{3}(7a_0 + 3)x^3 + \frac{1}{24}(43a_0 + 18)x^4 + \cdots. \]
Now $y(x) = z(x - 2)$, so

$$y(x) = a_0 + (1 + 2a_0)(x - 2) + \frac{1}{2}(5a_0 + 2)(x - 2)^2$$

$$+ \frac{1}{3}(7a_0 + 3)(x - 2)^3 + \frac{1}{24}(43a_0 + 18)(x - 2)^4 + \cdots.$$