WORKSHEET: COMPLEX NUMBERS

Names and student IDs: Solutions [πππ-πππ-ππππ]

Consider the complex numbers

\[ c_1 = 2 - 3i \quad \text{and} \quad c_2 = -3 + 4i \]

All answers to the first problems should be expressed in the form \( a + bi \) with \( a \) and \( b \) real (except for absolute values, which are automatically real).

1. Find \( c_1 + c_2 \).

Solution: \( c_1 + c_2 = (2 - 3i) + (-3 + 4i) = (2 + (-3)) + ((-3) + 4)i = -1 + i \).

2. Find \( c_1 - c_2 \).

Solution:
\[
 c_1 - c_2 = (2 - 3i) - (-3 + 4i) = (2 - (-3)) + ((-3) - 4)i = 5 - 7i. 
\]

3. Find \( -c_1 \).

Solution:
\[
 -c_1 = -(2 - 3i) = -2 + 3i. 
\]

4. Find \( c_1c_2 \).

Solution:
\[
 c_1c_2 = (2 - 3i)(-3 + 4i) \\
 = 2(-3) + (2)(4)i + (-3)(-3)i + (-3)(4)i^2 \\
 = -6 + 8i + 9i - 12i^2 = (-6 + 12) + (8 + 9)i = 6 + 17i. 
\]

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5. Find $c^{-1}_2$.

**Solution:**

$$c^{-1}_2 = \frac{1}{-3 + 4i} \left( \frac{-3 - 4i}{-3 - 4i} \right) = \frac{-3 - 4i}{(-3)^2 - (4i)^2} = \frac{-3 - 4i}{9 + 16} = -\frac{3}{25} - \frac{4}{25}i.$$  

6. Find $c_1/c_2$. (It is easiest to use the answer from the previous problem as an intermediate step.)

**Solution:**

$$\frac{c_1}{c_2} = (2 - 3i) \left( \frac{1}{-3 + 4i} \right) = (2 - 3i) \left( \frac{-3}{25} - \frac{4}{25}i \right)$$

$$= 2 \left( -\frac{3}{25} \right) + 2 \left( -\frac{4}{25}i \right) - 3 \left( -\frac{3}{25} \right)i - 3 \left( -\frac{4}{25} \right)i^2$$

$$= -\frac{6}{25} - \frac{8}{25}i + \frac{9}{25}i - \frac{12}{25} = -\frac{18}{25} + \frac{1}{25}i.$$  

7. Find $|c_1|$.

**Solution:**

$$|c_1| = |2 - 3i| = \sqrt{2^2 + (-3)^2} = \sqrt{13}.$$  

8. Find $\overline{c_2}$.

**Solution:**

$$\overline{c_2} = 3 + 4i = -3 - 4i.$$  

9. Check by a direct calculation that if $a, b, c, d$ are all real, then $$(a + bi) + (c + di) = a + bi + c + di.$$  

**Solution:**

$$(a + bi) + (c + di) = \overline{(a + c) + (b + d)i} = (a + c) - (b + d)i$$

$$= (a - bi) + (c - di) = a + bi + c + di.$$
10. Check by a direct calculation that if $a, b, c, d$ are all real, then $\left| (a + bi)(c + di) \right| = |a + bi||c + di|$.

**Solution:**

\[
\left| (a + bi)(c + di) \right|^2 = \left| (ac - bd) + (ad + bc)i \right|^2 = (ac - bd)^2 + (ad + bc)^2 \\
= a^2c^2 - 2abcd + b^2d^2 + a^2d^2 + 2adbc + b^2c^2 \\
= a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2,
\]

and

\[
\left[ |a + bi||c + di| \right]^2 = (a^2 + b^2)(c^2 + d^2) \\
= a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2.
\]

The right hand sides are equal. Since

\[
\left| (a + bi)(c + di) \right| \geq 0 \quad \text{and} \quad |a + bi||c + di| \geq 0,
\]

it follows that

\[
\left| (a + bi)(c + di) \right| = |a + bi||c + di|.
\]

Bonus problem. Use the quadratic formula to find two complex solutions, say $z_1$ and $z_2$, to the equation $z^2 + z + 1 = 0$. Then verify the following:

\[
z_1^2 = z_2, \quad z_2^2 = z_1, \quad z_1^3 = 1, \quad \text{and} \quad z_2^3 = 1.
\]

(Observe that now 1 has at least three cube roots, instead of just one.)

**Solution:** The roots are

\[
-\frac{1}{2} + \frac{\sqrt{3}}{2}i \quad \text{and} \quad -\frac{1}{2} - \frac{\sqrt{3}}{2}i.
\]

No solution has been written to the rest, but it doesn’t matter which you call $z_1$ and which you call $z_2$.

**Caution:** The relation between $z_1$ and $z_2$ is very special to the particular equation $z^2 + z + 1 = 0$. In general, if a quadratic equation with real coefficients has two nonreal roots $z_1$ and $z_2$, the only relation between them is that $z_2 = \bar{z}_1$ (and, of course, then $z_1 = \bar{z}_2$).