WORKSHEET: COMPLEX NUMBERS

Names and student IDs: Solutions

Consider the complex numbers

\[ c_1 = 2 - 3i \quad \text{and} \quad c_2 = -3 + 4i \]

All answers to the first problems should be expressed in the form \(a + bi\) with \(a\) and \(b\) real (except for absolute values, which are automatically real).

1. Find \(c_1 + c_2\).

\[c_1 + c_2 = (2 - 3i) + (-3 + 4i) = (2 + (-3)) + ((-3) + 4)i = -1 + i.\]

2. Find \(c_1 - c_2\).

\[c_1 - c_2 = (2 - 3i) - (-3 + 4i) = (2 - (-3)) + ((-3) - 4)i = 5 - 7i.\]

3. Find \(-c_1\).

\[-c_1 = -(2 - 3i) = -2 + 3i.\]

4. Find \(c_1 c_2\).

Date: 4 June 2018.
Solution:
\(c_1 + c_2 = (2 - 3i)(-3 + 4i)\)
\[= 2(-3) + (2)(4)i + (-3)(-3)i + (-3)(4)i^2\]
\[= -6 + 8i + 9i - 12i^2 = (-6 + 12) + (8 + 9)i = 6 + 17i.\]

5. Find \(c_2^{-1}\).

Solution:
\[c_2^{-1} = \frac{1}{-3 + 4i} \left( \frac{-3 - 4i}{-3 - 4i} \right) = \frac{-3 - 4i}{(-3)^2 - (4i)^2} = \frac{-3 - 4i}{9 + 16} = -\frac{3}{25} - \frac{4}{25}i.\]

6. Find \(c_1/c_2\). (It is easiest to use the answer from the previous problem as an intermediate step.)

Solution:
\[\frac{c_1}{c_2} = (2 - 3i) \left( \frac{1}{-3 + 4i} \right) = (2 - 3i) \left( \frac{-3}{25} - \frac{4}{25}i \right)\]
\[= 2 \left( -\frac{3}{25} \right) + 2 \left( -\frac{4}{25} \right)i - 3 \left( -\frac{3}{25} \right)i - 3 \left( -\frac{4}{25} \right)i^2\]
\[= -\frac{6}{25} - \frac{8}{25}i + \frac{9}{25}i = -\frac{18}{25} + \frac{1}{25}i.\]

7. Find \(|c_1|\).

Solution:
\[|c_1| = |2 - 3i| = \sqrt{2^2 + (-3)^2} = \sqrt{13}.\]

8. Find \(\overline{c_2}\).

Solution:
\[\overline{c_2} = 3 + 4i = -3 - 4i.\]

9. Check by a direct calculation that if \(a, b, c, d\) are all real, then \(\overline{(a + bi) + (c + di)} = \overline{a + bi} + \overline{c + di}\).
\begin{align*}
(a + bi) + (c + di) &= (a + c) + (b + d)i = (a + c) - (b + d)i \\
&= (a - bi) + (c - di) = a + bi + c + di.
\end{align*}

10. Check by a direct calculation that if \(a, b, c, d\) are all real, then 
\[|(a + bi)(c + di)| = |a + bi||c + di|.
\]

**Solution:**
\[|(a + bi)(c + di)|^2 = |(ac - bd) + (ad + bc)i|^2 = (ac - bd)^2 + (ad + bc)^2
\]
\[= a^2c^2 - 2acbd + b^2d^2 + a^2d^2 + 2adbc + b^2c^2
\]
\[= a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2,
\]
and
\[|a + bi||c + di|^2 = (a^2 + b^2)(c^2 + d^2)
\]
\[= a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2.
\]
The right hand sides are equal. Since
\[|(a + bi)(c + di)| \geq 0 \quad \text{and} \quad |a + bi||c + di| \geq 0,
\]
it follows that 
\[|(a + bi)(c + di)| = |a + bi||c + di|.
\]

**Bonus problem.** Use the quadratic formula to find two complex solutions, say \(z_1\) and \(z_2\), to the equation \(z^2 + z + 1 = 0\). Then verify the following:
\[z_1^2 = z_2, \quad z_2^2 = z_1, \quad z_1^3 = 1, \quad \text{and} \quad z_2^3 = 1.
\]
(Observe that now 1 has at least three cube roots, instead of just one.)

**Solution:** The roots are 
\[-\frac{1}{2} + \frac{\sqrt{3}}{2}i \quad \text{and} \quad -\frac{1}{2} - \frac{\sqrt{3}}{2}i.
\]
No solution has been written to the rest, but is doesn’t matter which you call \(z_1\) and which you call \(z_2\).

**Caution:** The relationship between \(z_1\) and \(z_2\) is very special to the particular equation \(z^2 + z + 1 = 0\). In general, if a quadratic equation
with real coefficients has two nonreal roots $z_1$ and $z_2$, the only relation between them is that $z_2 = \overline{z_1}$ (and, of course, then $z_1 = \overline{z_2}$).