Recall that we defined, for all complex numbers $z$,

\[ e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \cdots, \]

\[ \cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots, \quad \text{and} \quad \sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots, \]

and that these series do in fact converge for all complex numbers $z$.

Also recall that the usual laws of arithmetic hold for complex numbers, together with the usual properties of the absolute value function $z \mapsto |z|$. (However, it doesn’t make sense to say a complex number is positive or negative.)

1. Check that $i^3 = -i$, $i^4 = 1$, and $i^5 = i$. What is $i^{103}$?

2. By substituting $iz$ for $z$ in the series for $e^z$, verify that $e^{iz} = \cos(z) + i\sin(z)$ for any complex number $z$. 

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3. Consider the equation from Problem 2, but put a *real* number $y$ in place of $z$. One gets $e^{iy} = \cos(y) + i\sin(y)$ for all real numbers $y$. Use this equation to find all real numbers $y$ such that $e^{iy} = 1$.

Hint: $y = 0$ and $y = 2\pi$ are among the solutions, but there are no solutions in $(0, 2\pi)$. (Why not?)

4. Use the equation from Problem 3 to show that $|e^{iy}| = 1$ for all real numbers $y$.

5. Recall that $e^{w+z} = e^w e^z$ for all complex numbers $w$ and $z$. Use this and the formula in Problem 4 to show that if $a$ and $b$ are real, then $|e^{a+bi}| = e^a$.

6. Find all complex numbers $z$ such that $e^z = 1$. 
7. Find all complex numbers \( z \) such that \( e^z = -1 \).

8. By similar methods to Problem 2, verify that for any complex number \( z \), we have

\[
\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.
\]
9. Using the power series expansions, find a formula for \( \cos(z) \) which is similar to that in Problem 8.

If \( z = a + bi \) with \( a \) and \( b \) real, then \( a \) is called the real part of \( z \), written \( \text{Re}(z) \), and \( b \) is called the imaginary part of \( z \), written \( \text{Im}(z) \). (Note the convention: \( \text{Im}(z) \) is a real number. Also, \( z = \text{Re}(z) + \text{Im}(z)i \).)

Bonus problem 1. If \( \alpha \) is real and \( z \) is complex, express \( \text{Re}(\alpha z) \) and \( \text{Im}(\alpha z) \) in terms of \( \text{Re}(z) \), \( \text{Im}(z) \), and \( \alpha \).

Bonus problem 2. If \( w \) and \( z \) are complex, what are \( \text{Re}(w + z) \) and \( \text{Im}(w + z) \) in terms of the real and imaginary parts of \( w \) and \( z \)?

Bonus problem 3. If \( w \) and \( z \) are complex, what are \( \text{Re}(wz) \) and \( \text{Im}(wz) \) in terms of the real and imaginary parts of \( w \) and \( z \)? (They are not \( \text{Re}(w)\text{Re}(\bar{z}) \) and \( \text{Im}(w)\text{Im}(\bar{z}) \).)