At least 80% of the points on the real exam will be modifications of problems from the problems below, homework problems (particularly written homework), worksheet problems, and problems from the sample and real Midterms 0. Note, though, that the exact form of the functions to be expanded in series, limits and sums to be found, etc., could vary substantially, and the methods required to do them might occur in different combinations.

Be sure to get the notation right! (This is a frequent source of errors.) You have seen the correct notation in the book, in handouts, in files posted on the course website, and on the blackboard; use it. The right notation will help you get the mathematics right, and incorrect notation will lose points.

Here is the instruction sheet for Midterm 1:

1. DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO.
3. The following are all prohibited: Calculators (of any kind), cell phones, laptops, iPods, electronic dictionaries, and any other electronic devices or communication devices. All electronic or communication devices you have with you must be turned completely off and put inside something (pack, purse, etc.) and out of sight.
4. The point values are as indicated in each problem; total 100 points.
5. Write all answers on the test paper. Use the back of the page with the extra credit problems for long answers or scratch work.
6. Show enough of your work that your method is obvious. Be sure that every statement you write is correct. Cross out any material you do not wish to have considered. Correct answers with insufficient justification or accompanied by additional incorrect statements will not receive full credit. Correct guesses to problems requiring significant work, and correct answers obtained after a sequence of mostly incorrect steps, will receive no credit.
7. Unless otherwise specified, you can use the series expansions centered at \(x = 0\) for \(e^x\), \(\sin(x)\), \(\cos(x)\), and \((1 + x)^k\) (binomial series) without deriving them.
8. Be sure you say what you mean, and use correct notation. Credit will be based on what you say, not what you mean.
9. When exact values are specified, give answers such as \(\frac{1}{7}\), \(\sqrt{2}\), \(\ln(2)\), or \(\frac{2\pi}{9}\). Decimal approximations will not be accepted.
10. Final answers must always be simplified unless otherwise specified. (General principle: Combinations of powers and factorials need not be multiplied out, and “special” cancellations need not be made. Thus, in most problems, the expression \(3^7/7!\) is fine. However, \((n + 2)!/n!\) should be simplified to \((n + 1)(n + 2)\).)
11. Grading complaints must be submitted in writing at the beginning of the class period after the one in which the exam is returned (usually by the Tuesday after the exam).
12. Time: 50 minutes.
Problems.

1. (5 points/part.) For each of the following sequences \((a_n)_{n=1}^{\infty}\), find its limit (possibly \(\infty\) or \(-\infty\)), giving reasons, or explain why the sequence neither converges nor diverges to \(\infty\) or \(-\infty\).

[Note: Sequence convergence problems are overrepresented here, because there are many different kinds.]

a. \(a_n = \frac{n + 19}{19n^2 + 7}\) for strictly positive integers \(n\).

b. \(a_n = \frac{n^2 - 6n}{(3n + 11)^2}\) for strictly positive integers \(n\).

c. \(a_n = \frac{e^n + 17}{10e^n + 9}\) for strictly positive integers \(n\).

d. \((a_n)_{n=1}^{\infty} = \left(6, -\frac{9}{2}, \frac{27}{8}, -\frac{81}{32}, \frac{243}{128}, \ldots\right)\).

e. \(a_n = \left(-\sqrt{2}\right)^n\) for strictly positive integers \(n\).

f. \(a_n = \frac{n!}{5^{n+1}}\) for strictly positive integers \(n\).

g. \(a_n = \frac{(2n + 1)!}{(2n)!}\) for strictly positive integers \(n\).

h. \(a_n = \frac{(-1)^{n+1}(n + 19)}{19n + 7}\) for strictly positive integers \(n\).

i. \((a_n)_{n=1}^{\infty} = (1, -1, 1, 1, -1, 1, 1, 1, -1, 1, 1, -1, 1, \ldots)\).

j. \(a_n = \ln(n + 3)\) for strictly positive integers \(n\).

k. \(a_n = \arctan(n - 2)\) for strictly positive integers \(n\).

l. \(a_n = \frac{\ln(n)}{3n}\) for strictly positive integers \(n\).

m. \(a_n = \sin\left(\frac{n + 7}{7n + 22}\right)\) for strictly positive integers \(n\).

n. \(a_n = \frac{\cos(n^3 + 6)}{6n}\) for strictly positive integers \(n\).

2. (5 points.) A sequence \((a_n)_{n=1}^{\infty}\) is given by \((a_n)_{n=1}^{\infty} = \left(\frac{1}{5}, -\frac{2}{6}, \frac{3}{7}, -\frac{4}{8}, \frac{5}{9}, -\frac{6}{10}, \frac{7}{11}, \ldots\right)\).

Assuming the pattern continues, write down a formula for \(a_n\) for arbitrary \(n\).
3. (4 points.) A sequence \( (a_n)_{n=1}^{\infty} \) is given by \( (a_n)_{n=1}^{\infty} = \left(9, \frac{8}{3}, 4, 6, 9, \ldots\right) \).

Assuming the pattern continues, what is the next term?

4. (7 points/part.) For each of the following functions, find the Taylor polynomial of the given degree and centered at the given point.

   a. \( f(x) = e^x \), degree 7, centered at \( x = 1 \) (not \( x = 0 \)).

   b. \( f(x) = \cos(7x^2) \), degree 8, centered at \( x = 0 \).

   c. \( g(x) = \frac{1}{(1 + x)^{4/3}} \), degree 4, centered at \( x = 0 \).

   d. \( g(x) = (1 - 3x)^{1/4} \), degree 3, centered at \( x = 0 \).

   e. \( h(x) = \sin(2x) - \cos(3x) \), degree 6, centered at \( x = 0 \).

   f. \( h(x) = \cos(2x)e^x \), degree 4, centered at \( x = 0 \).

   g. \( f(x) = \sin(xe^x) \), degree 4, centered at \( x = 0 \).

   h. \( f(x) = \frac{6 + x}{2 + \sin(x)} \), degree 3, centered at \( x = 0 \).

5. (5 points.) Define \( f(x) = \cos(x^5) \) for all real \( x \). Find \( f^{(9)}(0) \). (Remember to show your work. Simplify your answer but don’t multiply out powers, factorials, etc.)

6. (5 points.) Define \( f(x) = \sin(3x^4) \) for all real \( x \). Find \( f^{(12)}(0) \). (Remember to show your work. Simplify your answer but don’t multiply out powers, factorials, etc.)

7. (6 points.) Let \( (a_n)_{n=1}^{\infty} \) be a sequence, and let \( L \) be a real number. State the precise definition of what it means to have \( \lim_{n \to \infty} a_n = L \).

8. (8 points.) Define a sequence \( (a_n)_{n=1}^{\infty} \) by \( a_n = \sqrt[n]{n} \) for \( n = 1, 2, \ldots \). For \( M = 137 \), find some integer \( N > 0 \) such that for all \( n > N \) we have \( a_n > M \), and show that your choice works. (You need not find the best value of \( N \).)

9. (8 points.) Define a sequence \( (x_n)_{n=1}^{\infty} \) by \( x_n = \frac{3n + 1}{n} \) for \( n = 1, 2, \ldots \). For \( \varepsilon = 0.2 \), find some integer \( N > 0 \) such that for all \( n > N \) we have \( |x_n - 3| < \varepsilon \), and show that your choice works. (You need not find the best value of \( N \).)
10. (8 points.) Define a sequence \((y_n)_{n=1}^\infty\) by \(y_n = \frac{1}{2^{n/2}}\) for \(n = 1, 2, \ldots\). For \(\varepsilon = \frac{1}{16}\), find some integer \(N > 0\) such that for all \(n > N\) we have \(|x_n| < \varepsilon\), and show that your choice works. (You need not find the best value of \(N\).)

11. (7 points.) Determine whether the series \(\sum_{n=1}^\infty \frac{7}{\sqrt{n^2 + 7}}\) converges. Be sure to show your reasoning.

12. (7 points.) Determine whether the series \(\sum_{n=1}^\infty \frac{12}{\sqrt{n^5 + 12}}\) converges. Be sure to show your reasoning.

13. (6 points.) Express the number with decimal expansion 0.009009009009\ldots as a ratio of integers.

14. (7 points/part.) For each of the following series, determine whether it is convergent. Be sure to show your reasoning. If the series is convergent, find its sum.
   a. \(\sum_{n=1}^\infty [\sin(4)]^n\)
   b. \(\sum_{n=1}^\infty \left(\frac{2}{3^n} - \frac{3}{4^n}\right)\)
   c. \(\sum_{n=1}^\infty \left(\frac{2}{n} - \frac{2}{n+3}\right)\)
   d. \(\sum_{n=0}^\infty \frac{2^{2n}}{3^{n+7}}\)
   e. \(\sum_{n=1}^\infty \frac{5}{n^2 + n}\)
   f. \(\sum_{n=1}^\infty \cos\left(\frac{1}{2n^2}\right)\)

15. (6 points/part.) For each of the following series, determine whether it is convergent. Be sure to show your reasoning.
a. $\sum_{n=1}^{\infty} \frac{1}{n^3 + 56n + 2}$

b. $\sum_{n=1}^{\infty} \left( \frac{2}{3^n} + \frac{4}{n^3} \right)$

c. $\sum_{n=1}^{\infty} \frac{\arctan(n) + 1}{n}$

d. $\sum_{n=1}^{\infty} \frac{7}{n^{7/6}}$