At least 80% of the points on the real exam will be modifications of problems from the problems below, homework problems (particularly written homework), worksheet problems, and problems from the sample and real Midterms 0. Note, though, that the exact form of the functions to be expanded in series, limits and sums to be found, etc., could vary substantially, and the methods required to do them might occur in different combinations.

Be sure to get the notation right! (This is a frequent source of errors.) You have seen the correct notation in the book, in handouts, in files posted on the course website, and on the blackboard; use it. The right notation will help you get the mathematics right, and incorrect notation will lose points.

Here is the instruction sheet for Midterm 1:

(1) DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO.
(2) Closed book, except for a 3 × 5 file card.
(3) The following are all prohibited: Calculators (of any kind), cell phones, laptops, iPods, electronic dictionaries, and any other electronic devices or communication devices. All electronic or communication devices you have with you must be turned completely off and put inside something (pack, purse, etc.) and out of sight.
(4) The point values are as indicated in each problem; total 100 points.
(5) Write all answers on the test paper. Use the back of the page with the extra credit problems for long answers or scratch work.
(6) Show enough of your work that your method is obvious. Be sure that every statement you write is correct. Cross out any material you do not wish to have considered. Correct answers with insufficient justification or accompanied by additional incorrect statements will not receive full credit. Correct guesses to problems requiring significant work, and correct answers obtained after a sequence of mostly incorrect steps, will receive no credit.
(7) Unless otherwise specified, you can use the series expansions centered at \( x = 0 \) for \( e^x \), \( \sin(x) \), \( \cos(x) \), and \( (1 + x)^k \) (binomial series) without deriving them.
(8) Be sure you say what you mean, and use correct notation. Credit will be based on what you say, not what you mean.
(9) When exact values are specified, give answers such as \( \frac{1}{\pi} \), \( \sqrt{2} \), \( \ln(2) \), or \( \frac{2\pi}{9} \). Decimal approximations will not be accepted.
(10) Final answers must always be simplified unless otherwise specified. (General principle: Combinations of powers and factorials need not be multiplied out, and “special” cancellations need not be made. Thus, in most problems, the expression \( 3^7/7! \) is fine. However, \( (n + 2)!/n! \) should be simplified to \( (n + 1)(n + 2). \))
(11) Grading complaints must be submitted in writing at the beginning of the class period after the one in which the exam is returned (usually by the Tuesday after the exam).
(12) Time: 50 minutes.
Problems.

1. (5 points/part.) For each of the following sequences \((a_n)_{n=1}^{\infty}\), find its limit (possibly \(\infty\) or \(-\infty\)), giving reasons, or explain why the sequence neither converges nor diverges to \(\infty\) or \(-\infty\). [Note: Sequence convergence problems are overrepresented here, because there are many different kinds.]

   a. \(a_n = \frac{n + 19}{19n^2 + 7}\) for strictly positive integers \(n\).

   Answer: \(\lim_{n \to \infty} a_n = 0\).

   b. \(a_n = \frac{n^2 - 6n}{(3n + 11)^2}\) for strictly positive integers \(n\).

   Answer: \(\lim_{n \to \infty} a_n = \frac{1}{9}\).

   c. \(a_n = \frac{e^n + 17}{17e^n + 9}\) for strictly positive integers \(n\).

   Solution: \(\lim_{n \to \infty} \frac{e^n - 17}{17e^n + 9} = \lim_{n \to \infty} \frac{e^{-n}(e^n - 17)}{e^{-n}(17e^n + 9)} = \lim_{n \to \infty} \frac{1 - 17e^{-n}}{17 + 9e^{-n}} = \frac{1 - 17 \cdot 0}{17 + 9 \cdot 0} = \frac{1}{17}\).

   d. \((a_n)_{n=1}^{\infty} = \left(6, -\frac{9}{2}, \frac{27}{8}, -\frac{81}{32}, \frac{243}{128}, \ldots\right)\).

   Solution: The general formula is \(a_n = 6 \left(-\frac{2}{3}\right)^{n-1}\) for \(n = 1, 2, 3, \ldots\). Since \(\left|\frac{2}{3}\right| = \frac{2}{3} < 1\), we have \(\lim_{n \to \infty} \left(-\frac{2}{3}\right)^{n-1} = 0\), so \(\lim_{n \to \infty} a_n = 0\).

   e. \(a_n = (-\sqrt{2})^n\) for strictly positive integers \(n\).

   Solution: Since \(\left|-\sqrt{2}\right| = \sqrt{2} > 1\), we have \(\lim_{n \to \infty} |a_n| = \lim_{n \to \infty} (\sqrt{2})^n = \infty\).

   However, the terms alternate in sign, with the even index terms going to \(\infty\) and the odd index terms going to \(-\infty\). So the sequence doesn’t diverge to \(\infty\) or \(-\infty\). Thus, all that can be said is that \(\lim_{n \to \infty} a_n\) does not exist.

   f. \(a_n = \frac{n!}{5n+1}\) for strictly positive integers \(n\).
Answer: \( \lim_{n \to \infty} a_n = \infty. \)

g. \( a_n = \frac{(2n + 1)!}{(2n)!} \) for strictly positive integers \( n. \)

**Solution (sketch):** We have

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} (2n + 1) = \infty.
\]

h. \( a_n = \frac{(-1)^{n+1}(n + 19)}{19n + 7} \) for strictly positive integers \( n. \)

**Answer:** \( \lim_{n \to \infty} a_n \) does not exist. The sequence doesn’t even diverge to \( \infty \) or \(-\infty\).

i. \((a_n)_{n=1}^\infty = (1, -1, 1, 1, -1, 1, 1, -1, 1, 1, 1, -1, 1, \ldots).\)

**Solution:** \( \lim_{n \to \infty} a_n \) does not exist. The sequence doesn’t even diverge to \( \infty \) or \(-\infty\).

j. \( a_n = \ln(n + 3) \) for strictly positive integers \( n. \)

**Solution:** \( \lim_{x \to \infty} \ln(x) = \infty \), so \( \lim_{x \to \infty} \ln(x + 3) = \infty \), so \( \lim_{n \to \infty} \ln(n + 3) = \infty. \)

k. \( a_n = \arctan(n - 2) \) for strictly positive integers \( n. \)

**Solution:** \( \lim_{x \to \infty} \arctan(x) = \frac{\pi}{2} \), so \( \lim_{x \to \infty} \arctan(x - 2) = \frac{\pi}{2} \), so \( \lim_{n \to \infty} \arctan(n - 2) = \frac{\pi}{2}. \)

l. \( a_n = \frac{\ln(n)}{3n} \) for strictly positive integers \( n. \)

**Solution (sketch):** Show that \( \lim_{x \to \infty} \frac{\ln(x)}{3x} = 0 \), for example by using L’Hopital’s Rule. So \( \lim_{n \to \infty} a_n = 0. \)

m. \( a_n = \sin \left( \frac{n + 7}{7n + 22} \right) \) for strictly positive integers \( n. \)

**Solution (sketch):** You know (details omitted) that

\[
\lim_{n \to \infty} \frac{n + 7}{7n + 22} = \frac{1}{7},
\]

Since \( x \mapsto \sin(x) \) is continuous at \( \frac{1}{7} \), it follows that

\[
\lim_{n \to \infty} \sin \left( \frac{n + 7}{7n + 22} \right) = \sin \left( \frac{1}{7} \right),
\]

n. \( a_n = \frac{\cos(n^3 + 6)}{6n} \) for strictly positive integers \( n. \)
Solution: For \( n = 1, 2, 3, \ldots \), we have \(-1 \leq \cos(n^3 + 6) \leq 1\), so
\[
-\frac{1}{6n} \leq \frac{\cos(n^3 + 6)}{6n} \leq \frac{1}{6n}.
\]
Now
\[
\lim_{n \to \infty} \left(-\frac{1}{6n}\right) = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{6n} = 0,
\]
so
\[
\lim_{n \to \infty} \frac{\cos(n^3 + 6)}{6n} = 0
\]
by the Squeeze Theorem.

2. (5 points.) A sequence \((a_n)_{n=1}^{\infty}\) is given by \((a_n)_{n=1}^{\infty} = \left(\frac{1}{5}, -\frac{2}{6}, \frac{3}{7}, -\frac{4}{8}, \frac{5}{9}, -\frac{6}{10}, \frac{7}{11}, \ldots\right)\).

Assuming the pattern continues, write down a formula for \(a_n\) for arbitrary \(n\).

Solution: \(a_n = \frac{(-1)^{n+1}(n + 1)}{n + 5}\) or \(a_n = \frac{(-1)^n(n + 1)}{n + 5}\).

3. (4 points.) A sequence \((a_n)_{n=1}^{\infty}\) is given by \((a_n)_{n=1}^{\infty} = \left(9, 6, 4, \frac{8}{3}, \ldots\right)\).

Assuming the pattern continues, what is the next term?

Solution: This sequence is a geometric sequence with common ratio \(2/3\), so the next term is \(\left(\frac{2}{3}\right)\left(\frac{8}{3}\right) = \frac{16}{9}\).

4. (7 points/part.) For each of the following functions, find the Taylor polynomial of the given degree and centered at the given point.

a. \(f(x) = e^x\), degree 7, centered at \(x = 1\) (not \(x = 0\)).

Solution: We use the formula
\[
p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots + \frac{f^{(n)}(0)}{n!} x^n
\]
for the \(n\)-th Taylor polynomial, with \(n = 6\). Ww have \(f'(x) = e^{x-1}\), \(f''(x) = e^{x-1}\), etc.; in fact, \(f^{(n)}(x) = e^{x-1}\) for all positive integers \(n\). So
\[
f(0) = e, \quad f'(0) = e, \quad f''(0) = e, \quad \cdots, \quad f^{(7)}(0) = e.
\]
Therefore the degree 6 Taylor polynomial is
\[
e + ex + \frac{e}{2!} x^2 + \frac{e}{3!} x^3 + \frac{e}{4!} x^4 + \frac{e}{5!} x^5 + \frac{e}{6!} x^6 + \frac{e}{7!} x^7.
\]
The answer
\[
q(x) = 1 + (x - 1) + \frac{1}{2!}(x - 1)^2 + \cdots + \frac{1}{7!}(x - 1)^7
\]
isn’t right. For example, the next term, \(\frac{1}{8!}(x - 1)^8\), contributes something of degree zero. Another way to see this is to check that
\[
q(0) = 1 + (-1) + \frac{1}{2!}(-1)^2 + \frac{1}{3!}(-1)^3 + \frac{1}{4!}(-1)^4 + \frac{1}{5!}(-1)^5 + \frac{1}{6!}(-1)^6 + \frac{1}{7!}(-1)^7 \neq e.
\]

b. \(f(x) = \cos(7x^2)\), degree 8, centered at \(x = 0\).

Solution (sketch): Substitute \(7x^2\) in the degree 4 Taylor polynomial for \(\cos(x)\) centered at 0, getting
\[
1 - \frac{7^2}{2!}x^4 + \frac{7^4}{4!}x^8.
\]

c. \(g(x) = \frac{1}{(1 + x)^{4/3}}\), degree 4, centered at \(x = 0\).

Solution (sketch): Use the binomial series, getting
\[
1 + \left(\frac{4}{3}\right) x + \left(\frac{4}{3}\right) \left(\frac{1}{3}\right) x^2 + \left(\frac{4}{3}\right) \left(\frac{1}{3}\right) \left(-\frac{2}{3}\right) x^3 + \left(\frac{4}{3}\right) \left(\frac{1}{3}\right) \left(-\frac{2}{3}\right) \left(-\frac{5}{3}\right) x^4
\]
\[
= 1 + \left(\frac{4}{3}\right) x + \left(\frac{4 \cdot 1}{3^2}\right) x^2 - \left(\frac{4 \cdot 1 \cdot 2}{3^3}\right) x^3 + \left(\frac{4 \cdot 1 \cdot 2 \cdot 5}{3^4}\right) x^4.
\]

d. \(g(x) = (1 - 3x)^{1/4}\), degree 3, centered at \(x = 0\).

Solution (sketch): Substitute \(-3x\) in the binomial series for \((1 + x)^{1/4}\), getting
\[
1 + \left(\frac{1}{4}\right) (-3x) + \left(\frac{1}{4}\right) \left(-\frac{3}{4}\right) (-3x)^2 + \left(\frac{1}{4}\right) \left(-\frac{3}{4}\right) \left(-\frac{7}{4}\right) (-3x)^3
\]
\[
= 1 - \left(\frac{3}{4}\right) x - \left(\frac{3^2 \cdot 1 \cdot 3}{4^2}\right) x^2 - \left(\frac{3^3 \cdot 1 \cdot 3 \cdot 7}{4^3}\right) x^3.
\]

e. \(h(x) = \sin(2x) - \cos(3x)\), degree 6, centered at \(x = 0\).

Solution (sketch): Substitute \(2x\) in the degree 6 Taylor polynomial for \(\sin(x)\) centered at 0, substitute \(3x\) in the degree 6 Taylor polynomial for \(\cos(x)\) centered at 0, and add the results, getting
\[
1 + 3x - \frac{2^2}{2!}x^2 - \frac{3^2}{3!}x^3 + \frac{2^4}{4!}x^4 + \frac{3^5}{5!}x^5 - \frac{2^6}{6!}x^6.
\]

f. \(h(x) = \cos(2x)e^x\), degree 4, centered at \(x = 0\).
Solution (sketch): Substitute $2x$ in the degree 4 Taylor polynomial for $\cos(x)$ centered at 0, multiply by the degree 4 Taylor polynomial for $e^x$ centered at 0, and drop all terms in the product in which the power of $x$ is 5 or more. The result is

$$1 + x - \frac{3x^2}{2} - \frac{11x^3}{6} - \frac{7x^4}{24}.$$ 

g. $f(x) = \sin(xe^x)$, degree 4, centered at $x = 0$.

Solution (sketch): The degree 4 Taylor polynomial for $xe^x$ centered at 0 is

$$x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!}.$$ 

Substitute this in the degree 4 Taylor polynomial for $\sin(x)$, expand the result, and drop all terms in the product in which the power of $x$ is 5 or more. The result is

$$x + x^2 + \frac{x^3}{3} - \frac{x^4}{3}.$$ 

h. $f(x) = \frac{6 + x}{2 + \sin(x)}$ degree 3, centered at $x = 0$.

Solution (sketch): The degree 3 Taylor polynomial for $6 + \sin(x)$ is $6 + x - \frac{x^3}{6}$. Carry out long division of $6 + x$ by this polynomial, ignoring everything in which the power of $x$ is 4 or more. The result is

$$3 - x + \frac{x^2}{2}.$$ 

5. (5 points.) Define $f(x) = \cos(x^5)$ for all real $x$. Find $f^{(9)}(0)$. (Remember to show your work. Simplify your answer but don’t multiply out powers, factorials, etc.)

Solution: Substitute $x^5$ in the degree 2 Taylor polynomial for $\cos(x)$. (This degree is chosen so that $2 \cdot 5 \geq 9$.) This gives the degree 10 Taylor polynomial for $\cos(x^5)$:

$$\cos(x^5) \approx 1 - \frac{x^{10}}{2!}.$$ 

Select the coefficient of $x^9$ and multiply it by $9!$, getting 0. So $f^{(9)}(0) = 0$.

6. (5 points.) Define $f(x) = \sin(3x^4)$ for all real $x$. Find $f^{(12)}(0)$. (Remember to show your work. Simplify your answer but don’t multiply out powers, factorials, etc.)

Solution: Substitute $3x^4$ in the degree 2 Taylor polynomial for $\sin(x)$. (This degree is chosen so that $3 \cdot 4 \geq 12$.) This gives the degree 10 Taylor polynomial for $\sin(3x^4)$:

$$\sin(x^5) \approx 3x^4 - \frac{(3x^4)^3}{3!} = 3x^4 - \frac{3^3}{3!}x^{12}.$$
Select the coefficient of $x^9$ and multiply it by $12!$, getting

$$f^{(12)}(0) = 12! \binom{3^3}{3!} = \frac{3^3 \cdot 12!}{3!}.$$

7. (6 points.) Let $(a_n)_{n=1}^\infty$ be a sequence, and let $L$ be a real number. State the precise definition of what it means to have $\lim_{n \to \infty} a_n = L$.

**Solution:** For every $\varepsilon > 0$ there is a positive integer $N$ such that for every integer $n > N$, we have $|a_n - L| < \varepsilon$.

8. (8 points.) Define a sequence $(a_n)_{n=1}^\infty$ by $a_n = \sqrt[3]{n}$ for $n = 1, 2, \ldots$. For $M = 137$, find some integer $N > 0$ such that for all $n > N$ we have $a_n > M$, and show that your choice works. (You need *not* find the best value of $N$.)

**Solution:** I am going to choose $N = 8,000,000$. (This is *not* the best choice.) Consider an arbitrary integer $n$ such that $n > 8,000,000$. Then

$$a_n = \sqrt[3]{n} > \sqrt[3]{8,000,000} = 200.$$

In particular, $a_n > 137$. So $N = 8,000,000$ works.

(For reference, the smallest choice which works is $N = 137^3$.)

9. (8 points.) Define a sequence $(x_n)_{n=1}^\infty$ by $x_n = \frac{3n + 1}{n}$ for $n = 1, 2, \ldots$. For $\varepsilon = 0.2$, find some integer $N > 0$ such that for all $n > N$ we have $|x_n - 3| < \varepsilon$, and show that your choice works. (You need *not* find the best value of $N$.)

**Scratchwork (not needed as part of the solution):** Rewrite

$$x_n = 3 + \frac{1}{n}$$

for $n = 1, 2, \ldots$. So

$$|x_n - 3| = \left| \frac{1}{n} \right| = \frac{1}{n}.$$

We want

$$\frac{1}{n} < 0.2 = \frac{1}{5}.$$

This is the same as $n > 5$.

**Solution:** We take $N = 5$. Consider an arbitrary integer $n$ such that $n > 5$. Then

$$|x_n - 3| = \left| \frac{3n + 1}{n} - 3 \right| = \frac{1}{n} < \frac{1}{5} = 0.2.$$

In particular, $|x_n - 3| < 0.2$. So $N = 5$ works.

(This actually is the best choice of $N$.)
10. (8 points.) Define a sequence \((y_n)_{n=1}^\infty\) by \(y_n = \frac{1}{2^{n/2}}\) for \(n = 1, 2, \ldots\). For \(\varepsilon = \frac{1}{16}\), find some integer \(N > 0\) such that for all \(n > N\) we have \(|x_n| < \varepsilon\), and show that your choice works. (You need not find the best value of \(N\).)

**Solution:** Scratchwork (not needed as part of the solution): We want \(\frac{1}{2^{n/2}} < \frac{1}{16}\). This happens if \(2^{n/2} > 16\), or \(2^n > 16^2 = 2^8\). So we need \(n > 8\).

**Solution:** I am going to choose \(N = 10\). (This is not the best choice.) Consider an arbitrary integer \(n\) such that \(n > 10\). Then

\[
|y_n - 0| = \left| \frac{1}{2^{n/2}} - 0 \right| = \frac{1}{2^{n/2}} < \frac{1}{2^{10/2}} = \frac{1}{2^5} = \frac{1}{32} \leq \frac{1}{16}.
\]

In particular, \(|y_n - 0| < 1/16\). So \(N = 10\) works.

(For reference, the smallest choice which works is \(N = 8\).)

11. (7 points.) Determine whether the series \(\sum_{n=1}^\infty \frac{7}{\sqrt{n^2 + 7}}\) converges. Be sure to show your reasoning.

**Solution:** For \(n = 1, 2, 3, \ldots\), we have

\[
0 < n^2 + 7 < n^2 + 7n + 2 = 8n^2.
\]

So

\[
0 < \sqrt{n^2 + 7} \leq \sqrt{8n^2} = 2n^{2/3},
\]

and

\[
0 < \frac{1}{\sqrt{n^2 + 7}} \leq \frac{1}{2n^{2/3}}.
\]

The series \(\sum_{n=0}^\infty \frac{1}{n^{2/3}}\) is divergent, since the exponent is \(\frac{2}{3} \leq 1\). So the series \(\sum_{n=0}^\infty \frac{1}{2n^{2/3}}\) is divergent.

The Comparison Test now implies that \(\sum_{n=0}^\infty \frac{1}{\sqrt{n^2 + 7}}\) is divergent.

Caution: it is also true that for \(n = 0, 1, 2, \ldots\), we have

\[
0 < \frac{1}{\sqrt{n^2 + 7}} \leq \frac{1}{n^{2/3}},
\]

and that \(\sum_{n=0}^\infty \frac{1}{n^{2/3}}\) is divergent. However, this tells you nothing about divergence of \(\sum_{n=0}^\infty \frac{\arctan(n) + 1}{n}\).

Be sure to get the inequalities right!
12. (7 points.) Determine whether the series \( \sum_{n=1}^{\infty} \frac{12}{\sqrt{n^3 + 12}} \) converges. Be sure to show your reasoning.

*Solution:* For \( n = 0, 1, 2, \ldots \), we have \( 0 \leq n^3 \leq n^3 + 12 \), so
\[
0 < n^{3/2} \leq \sqrt{n^3 + 12},
\]
so
\[
0 < \frac{1}{\sqrt{n^3 + 12}} \leq \frac{1}{n^{3/2}}.
\]
The series \( \sum_{n=0}^{\infty} \frac{1}{n^{3/2}} \) is convergent because the exponent \( \frac{3}{2} \) is greater than 1. So the Comparison Test implies that \( \sum_{n=0}^{\infty} \frac{1}{\sqrt{n^3 + 12}} \) is convergent.

*Caution:* it is also true that for \( n = 3, 4, 5, \ldots \), we have
\[
0 < \frac{1}{2 \cdot n^{3/2}} \leq \frac{1}{\sqrt{n^3 + 12}},
\]
and that \( \sum_{n=0}^{\infty} \frac{1}{2 \cdot n^{3/2}} \) is convergent. However, this tells you nothing about convergence of \( \sum_{n=0}^{\infty} \frac{1}{\sqrt{n^3 + 12}} \). Be sure to get the inequalities right!

13. (6 points.) Express the number with decimal expansion \( 0.009009009009 \ldots \) as a ratio of integers.

*Solution:* We write, using the formula for the sum of a geometric series at the third step,
\[
0.009009009009 \ldots = \sum_{n=1}^{\infty} \frac{9}{1000^n} = \frac{9}{1000} \sum_{m=0}^{\infty} \frac{1}{1000^m} = \left( \frac{9}{1000} \right) \left( \frac{1}{1 - \frac{1}{1000}} \right) = \left( \frac{9}{1000} \right) \left( \frac{1000}{999} \right) = \frac{9}{999} = \frac{1}{111}.
\]

14. (7 points/part.) For each of the following series, determine whether it is convergent. Be sure to show your reasoning. If the series is convergent, find its sum.

a. \( \sum_{n=1}^{\infty} [\sin(4)]^n \)
Solution: The series is a geometric series whose common ratio \( \sin(4) \) satisfies \(-1 < \sin(4) < 1\).

So

\[
\sum_{n=1}^{\infty} [\sin(4)]^n = [\sin(4)] \sum_{m=0}^{\infty} [\sin(4)]^m = \frac{\sin(4)}{1 - \sin(4)}.
\]

b. \( \sum_{n=1}^{\infty} \left( \frac{2}{3^n} - \frac{3}{4^n} \right) \)

Solution: The series \( \sum_{n=1}^{\infty} \frac{2}{3^n} \) is a geometric series whose common ratio \( \frac{1}{3} \) satisfies \(-1 < \frac{1}{3} < 1\).

So

\[
\sum_{n=1}^{\infty} \frac{2}{3^n} = \frac{2}{3} \sum_{m=0}^{\infty} \left( \frac{1}{3} \right)^m = \left( \frac{2}{3} \right) \left( \frac{1}{1 - \frac{1}{3}} \right) = 1.
\]

The series \( \sum_{n=1}^{\infty} \frac{3}{4^n} \) is a geometric series whose common ratio \( \frac{1}{4} \) satisfies \(-1 < \frac{1}{4} < 1\). So

\[
\sum_{n=1}^{\infty} \frac{3}{4^n} = \frac{3}{4} \sum_{m=0}^{\infty} \left( \frac{1}{4} \right)^m = \left( \frac{3}{4} \right) \left( \frac{1}{1 - \frac{1}{4}} \right) = 1.
\]

So

\[
\sum_{n=1}^{\infty} \left( \frac{2}{3^n} - \frac{3}{4^n} \right) = \sum_{n=1}^{\infty} \frac{2}{3^n} + \sum_{n=1}^{\infty} \frac{3}{4^n} = 1 + 1 = 2.
\]

c. \( \sum_{n=1}^{\infty} \left( \frac{2}{n} - \frac{2}{n+3} \right) \)

Solution (sketch): The partial sums are telescoping sums: for \( n \) large enough,

\[
\sum_{k=1}^{n} \left( \frac{2}{k} - \frac{2}{k+3} \right) = 2 + \frac{2}{2} + \frac{2}{3} - \frac{2}{n+1} - \frac{2}{n+2} - \frac{2}{n+3}.
\]

So

\[
\sum_{n=1}^{\infty} \left( \frac{2}{n} - \frac{2}{n+3} \right) = \lim_{n \to \infty} \left( 2 + \frac{2}{2} + \frac{2}{3} - \frac{2}{n+1} - \frac{2}{n+2} - \frac{2}{n+3} \right) = 2 + \frac{2}{2} + \frac{2}{3} = \frac{11}{3}.
\]

d. \( \sum_{n=0}^{\infty} \frac{2^{2n}}{3^{n+7}} \)

Solution: Rewrite

\[
\sum_{n=0}^{\infty} \frac{2^{2n}}{3^{n+7}} = \sum_{n=0}^{\infty} \frac{4^n}{3^{n+7}} = \sum_{n=0}^{\infty} \left( \frac{1}{3^7} \right) \left( \frac{4}{3} \right)^n.
\]

This is a geometric series whose common ratio \( \frac{4}{3} \) satisfies \( \frac{4}{3} < 1 \). Therefore it is divergent.
e. $\sum_{n=1}^{\infty} \frac{5}{n^2 + n}$

Solution: Use partial fractions to write

$$\sum_{n=1}^{\infty} \frac{5}{n^2 + n} = \sum_{n=1}^{\infty} \left( \frac{5}{n} - \frac{5}{n+1} \right).$$

The partial sums are telescoping sums:

$$\lim_{n \to \infty} \sum_{k=1}^{n} \left( \frac{5}{k} - \frac{5}{k+1} \right) = \lim_{n \to \infty} \left( 5 - \frac{5}{n+1} \right) = 5.$$

f. $\sum_{n=1}^{\infty} \cos \left( \frac{1}{2n^2} \right)$

Solution (sketch): $\lim_{n \to \infty} \cos \left( \frac{1}{2n^2} \right) = 1 \neq 0$, so $\sum_{n=1}^{\infty} \cos \left( \frac{1}{2n^2} \right)$ is divergent.

15. (6 points/part.) For each of the following series, determine whether it is convergent. Be sure to show your reasoning.

a. $\sum_{n=1}^{\infty} \frac{1}{n^3 + 56n + 2}$

Solution: For $n = 0, 1, 2, \ldots$, we have

$$0 < n^3 \leq n^3 + 56n + 2,$$

so

$$0 < \frac{1}{n^3 + 56n + 2} \leq \frac{1}{n^3}.$$

Since the exponent 3 in $\sum_{n=0}^{\infty} \frac{1}{n^3}$ is greater than 1, this series is convergent. So the Comparison Test implies that $\sum_{n=0}^{\infty} \frac{1}{n^3 + 56n + 2}$ is convergent.

Caution: it is also true that for $n \geq 112$, we have

$$0 < \frac{1}{2 \cdot n^3} \leq \frac{1}{n^3 + 56n + 2},$$

and that $\sum_{n=0}^{\infty} \frac{1}{2 \cdot n^3}$ is convergent. However, this tells you nothing about convergence of $\sum_{n=0}^{\infty} \frac{1}{n^3 + 56n + 2}$. Be sure to get the inequalities right!
b. \( \sum_{n=1}^{\infty} \left( \frac{2}{3^n} + \frac{4}{n^3} \right) \)

Solution: The series \( \sum_{n=1}^{\infty} \frac{1}{n^3} \) is convergent because the exponent 3 is greater than 1. Therefore the series \( \sum_{n=1}^{\infty} \frac{4}{n^3} \), being a constant multiple of a convergent series, is also convergent.

The series \( \sum_{n=1}^{\infty} \frac{2}{3^n} \) is convergent because it is a geometric series whose common ratio \( \frac{1}{3} \) satisfies \(-1 < \frac{1}{3} < 1\).

Thus the series \( \sum_{n=1}^{\infty} \left( \frac{2}{3^n} + \frac{4}{n^3} \right) \) is the sum of two convergent series, and is therefore convergent.

c. \( \sum_{n=1}^{\infty} \frac{\arctan(n) + 1}{n} \)

Solution: For \( n = 1, 2, 3, \ldots \), we have

\[ 0 < 1 \leq \arctan(n) + 1. \]

So

\[ 0 < \frac{1}{n} \leq \frac{\arctan(n) + 1}{n}. \]

The series \( \sum_{n=0}^{\infty} \frac{1}{n} \) is divergent. (The exponent is 1.) So the Comparison Test implies that \( \sum_{n=0}^{\infty} \frac{1}{n + \sqrt{n}} \) is divergent.

Caution: it is also true that for \( n = 0, 1, 2, \ldots \), we have

\[ 0 < \frac{\arctan(n) + 1}{n} \leq \frac{\pi}{2} + 1. \]

and that \( \sum_{n=0}^{\infty} \frac{\pi}{2} + \frac{1}{n} \) is divergent. However, this tells you nothing about divergence of \( \sum_{n=0}^{\infty} \frac{\arctan(n) + 1}{n} \).

Be sure to get the inequalities right!

d. \( \sum_{n=1}^{\infty} \frac{7}{n^{7/6}} \)
Solution: The series \( \sum_{n=1}^{\infty} \frac{1}{n^{7/6}} \) is convergent because the exponent \( \frac{7}{6} \) is greater than 1. Therefore the series \( \sum_{n=1}^{\infty} \frac{7}{n^{7/6}} \), being a constant multiple of a convergent series, is also convergent.