Notes. Achilles was a warrior of Greek myths from about 2500 years ago. His name is still in use today: the Achilles tendon. The speed he runs in these problems corresponds roughly to a four minute mile. The speeds attributed to the tortoise and the hare are entirely made up.

Zeno was apparently responsible for several related paradoxes, but the one in the second problem is the most famous.

1. (This is just to set the stage for Problem 2.) Achilles runs at 400 meters per minute. He races a hare. He gives the hare a head start: it starts 400 meters beyond the official starting point. The hare also runs at 400 meters per minute.

After one minute, Achilles has reached the place the hare started. But he has not caught up to the hare, which has gone another 400 meters in that time, and is now 800 meters from the starting point.

After $1 + 1 = 2$ minutes, Achilles has reached the place the hare got to after one minute. But he has up to not caught the hare, which has has gone another 400 meters in that time, and is now 1200 meters from the starting point.

After $1 + 1 + 1 = 3$ minutes, Achilles has reached the place the hare got to after 2 minutes. But he still has not caught up to the hare, which has has gone yet another 400 meters in that time.

Does Achilles ever catch up to the hare? ____

2. Achilles runs at 400 meters per minute (just as before). This time, he races a tortoise. He gives the tortoise the same head start as before: it starts 400 meters beyond the official starting point. The tortoise walks at 40 meters per minute.

(a) How long does it take Achilles to get to where the tortoise started? ____ minutes. At this time, how far beyond the starting point is the tortoise? ____ + ____ = ____ meters. How far beyond the starting point is Achilles? ____ meters.

(b) After the time in Part (a), how much longer does it take Achilles to get to where the tortoise was at that time? ____ minutes. How long since the beginning of the race did it take Achilles to get to this location? ____ + ____ = ____ minutes. At this new time, where is the tortoise? ____ meters. Where is Achilles? ____ meters.

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(c) After the time in Part (b), how much longer does it take Achilles to get to where the tortoise was at the time in Part (b)? _____ minutes. How long since the beginning of the race did it take Achilles to get to this location? _____ + _____ + _____ = _____ minutes. At this new time, where is the tortoise? _____ meters. Where is Achilles? _____ meters.

(d) Zeno argued as follows. By the time Achilles gets to where the tortoise started (which took one minute), the tortoise has gone 40 meters further. It takes Achilles 0.1 minutes to get to where the tortoise was after one minute, and by that time the tortoise has gone another 4 meters, so is still ahead of Achilles. By the time Achilles has covered that distance of 4 meters, the tortoise has gone another 0.4 meters, so is still ahead of Achilles. Etc. Therefore Achilles never catches up to the tortoise.

Do you believe this? Discuss in your group. Does Achilles ever catch up to the tortoise? If so, when do you think this happens?

3. (This problem uses almost the same numbers as in Problem 2, but in a different story.)

You start walking from the center of campus towards the McKenzie Pass (about 80 miles east of Eugene by road). In the first hour, you go 4 miles. In the second hour, you are tired, and go only 0.4 miles. In the third hour, you are much more tired, and go only 0.04 miles. In the fourth hour, you go only 0.004 miles. Etc.

(a) How far have you gone after one hour? _____ miles.

(b) How far have you gone after 2 hours? _____ + _____ = _____ miles.

(c) How far have you gone after 3 hours?
_____ + _____ + _____ = _____ miles.

(d) How far have you gone after 4 hours?
_____ + _____ + _____ + _____ = _____ miles.

(e) How far have you gone after 10 hours? _____ miles. (Use a calculator.)

(f) How far do you think you get in infinite time?)
4. (This problem uses the same story as Problem 3, but with different numbers.)
You start walking from the center of campus towards the McKenzie Pass. In the first hour, you go 1 mile. In the second hour, you are tired, and go only 1/2 miles. In the third hour, you are more tired, and go only 1/3 miles. In the fourth hour, you go only 1/4 miles. Etc.

(a) How far have you gone after one hour? _____ miles.

(b) How far have you gone after 2 hours? _____ + _____ = _____ miles.

(c) How far have you gone after 3 hours?
_____ + _____ + _____ = _____ miles.

(d) How far have you gone after 4 hours?
_____ + _____ + _____ + _____ = _____ miles.

(e) How far have you gone after 10 hours? _____ miles. (Use a calculator.)

(f) If you have a programmable calculator, use it to find out how far you have gone after:
1,000 hours: _____ miles;
1,000,000 hours: _____ miles;
1,000,000,000 hours: _____ miles;
1,000,000,000,000 hours: _____ miles.

(g) Do you think you will ever get to McKenzie Pass?

(h) Consider the following estimation:
\[1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}\right) + \cdots \]
\[\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}\right) + \cdots \]
\[= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \]
Now do you think you will ever get to McKenzie Pass?

(i) (Bonus.) Use a programmable calculator to redo the last calculation in part (f) in the reverse order:
\[\frac{1}{1,000,000,000,000} + \frac{1}{999,999,999,998} + \frac{1}{999,999,999,999} + \cdots + \frac{1}{2} + 1.\]
What do you get?