Notes. Achilles was a warrior of Greek myths from about 2500 years ago. His name is still in use today: the Achilles tendon. The speed he runs in these problems corresponds roughly to a four minute mile. The speeds attributed to the tortoise and the hare are entirely made up.

Zeno was apparently responsible for several related paradoxes, but the one in the second problem is the most famous.

1. (This is just to set the stage for Problem 2.) Achilles runs at 400 meters per minute. He races a hare. He gives the hare a head start: it starts 400 meters beyond the official starting point. The hare also runs at 400 meters per minute.

After one minute, Achilles has reached the place the hare started. But he has not caught up to the hare, which has gone another 400 meters in that time, and is now 800 meters from the starting point.

After 1 + 1 = 2 minutes, Achilles has reached the place the hare got to after one minute. But he has not caught the hare, which has gone another 400 meters in that time, and is now 1200 meters from the starting point.

After 1 + 1 + 1 = 3 minutes, Achilles has reached the place the hare got to after 2 minutes. But he still has not caught up to the hare, which has gone yet another 400 meters in that time.

Does Achilles ever catch up to the hare? No.

Achilles is always 400 meters behind the hare.
It takes Achilles:
- 1 minute to get to where the hare started.
- 1 + 1 = 2 minutes to get to where the hare was after 1 minute.
- 1 + 1 + 1 = 3 minutes to get to where the hare was after 2 minutes.

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2. Achilles runs at 400 meters per minute (just as before). This time, he races a tortoise. He gives the tortoise the same head start as before: it starts 400 meters beyond the official starting point. The tortoise walks at 40 meters per minute.

(a) How long does it take Achilles to get to where the tortoise started? 1 minutes. At this time, how far beyond the starting point is the tortoise? $400 + 40 = 440$ meters. How far beyond the starting point is Achilles? 400 meters.

(b) After the time in Part (a), how much longer does it take Achilles to get to where the tortoise was at that time? 0.1 minutes. How long since the beginning of the race did it take Achilles to get to this location? $1 + 0.1 = 1.1$ minutes. At this new time, where is the tortoise? $440 + 4 = 444$ meters. Where is Achilles? 440 meters.

(c) After the time in Part (b), how much longer does it take Achilles to get to where the tortoise was at the time in Part (b)? 0.01 minutes. How long since the beginning of the race did it take Achilles to get to this location? $1 + 0.1 + 0.01 = 1.11$ minutes. At this new time, where is the tortoise? $444 + 0.4 = 444.4$ meters. Where is Achilles? 444 meters.

(d) Zeno argued as follows. By the time Achilles gets to where the tortoise started (which took one minute), the tortoise has gone 40 meters further. It takes Achilles 0.1 minutes to get to where the tortoise was after one minute, and by that time the tortoise has gone another 4 meters, so is still ahead of Achilles. By the time Achilles has covered that distance of 4 meters, the tortoise has gone another 0.4 meters, so is still ahead of Achilles. Etc. Therefore Achilles never catches up to the tortoise.

Do you believe this? Discuss in your group. Does Achilles ever catch up to the tortoise? If so, when do you think this happens?

Solution: You can algebraically determine when Achilles catches up to the tortoise as follows. At time $t$ minutes after the race starts,
Achilles is 400\(t\) meters from the start, and the tortoise is \(400 + 40t\) meters from the start. Equate these and solve for \(t\):

\[
400 + 40t = 400t \\
400 = 400t - 40t = 360t \\
t = \frac{400}{360} = \frac{10}{9}.
\]

So Achilles catches up to the tortoise \(\frac{10}{9}\) minutes after the race starts.

Looking at the results in the previous parts, one should expect Achilles to catch up to the tortoise after

\[
1 + 0.1 + 0.01 + 0.001 + 0.0001 + 0.00001 + \cdots
\]

minutes. It seems reasonable that this sum should be the infinite decimal \(1.1111\ldots\), which you might recognize as the decimal expansion of \(\frac{10}{9}\).

Zeno’s claim amounts to saying that the sum of infinitely many strictly positive numbers must be infinite. This isn’t always correct (although sometimes it is true). Much of this course is about understanding this phenomenon, and deciding in specific cases which alternative is true.

3. (This problem uses almost the same numbers as in Problem 2, but in a different story.)

You start walking from the center of campus towards the McKenzie Pass (about 80 miles east of Eugene by road). In the first hour, you go 4 miles. In the second hour, you are tired, and go only 0.4 miles. In the third hour, you are much more tired, and go only 0.04 miles. In the fourth hour, you go only 0.004 miles. Etc.

(a) How far have you gone after one hour? 4 miles.

(b) How far have you gone after 2 hours? \(4 + 0.4 = 4.4\) miles.

(c) How far have you gone after 3 hours? 
\(4 + 0.4 + 0.04 = 4.44\) miles.

(d) How far have you gone after 4 hours? 
\(4 + 0.4 + 0.04 + 0.004 = 4.444\) miles.

(e) How far have you gone after 10 hours? \(4.444444444\) miles. (Use a calculator.)
(f) How far do you think you get in infinite time?

Solution: The answer certainly appears to have the infinite decimal expansion 4.4444... miles. This is 40/9. It certainly seems clear that you never got more that 5 miles from your starting point.

4. (This problem uses the same story as Problem 3, but with different numbers.)
You start walking from the center of campus towards the McKenzie Pass. In the first hour, you go 1 mile. In the second hour, you are tired, and go only 1/2 miles. In the third hour, you are more tired, and go only 1/3 miles. In the fourth hour, you go only 1/4 miles. Etc.

(a) How far have you gone after one hour? 1 miles.

(b) How far have you gone after 2 hours? $1 + \frac{1}{2} = \frac{3}{2}$ miles. (This is 1.5 miles.)

(c) How far have you gone after 3 hours?
$1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$ miles. (This about 1.8333 miles.)

(d) How far have you gone after 4 hours?
$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$ miles. (This about 2.08333 miles.)

(e) How far have you gone after 10 hours? about 2.928968 miles. (Use a calculator.)

(f) If you have a programmable calculator, use it to find out how far you have gone after:
1,000 hours: _____ miles;
1,000,000 hours: _____ miles;
1,000,000,000 hours: _____ miles;
1,000,000,000,000 hours: _____ miles.

Solution: These numbers seem to be mostly too big for a calculator to do in reasonable time. I borrowed a TI-84 Plus CE calculator from the math department. It gave the following (of course approximate) results:
I will have some computer results eventually, and update this solution sheet. With calculator numerical precision, one probably never gets more than about 23 or 24.

(g) Do you think you will ever get to McKenzie Pass?

*Solution:* With the numbers in the table, maybe. The computer results (to be posted later) suggest that the answer is no.

(h) Consider the following estimation:

\[
1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}\right) + \cdots
\]

\[
\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}\right) + \cdots
\]

\[
= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots
\]

Now do you think you will ever get to McKenzie Pass?

*Solution:* After 1 hour, you have gone 1 mile.

After 2 hours, you have gone 1 + 1/2 miles.

After 4 hours, you have gone more than 1 + 1/2 + 1/2 miles.

After 8 hours, you have gone more than 1 + 1/2 + 1/2 + 1/2 miles.

So after 2^n hours, you have gone more than (n + 1)/2 miles.

Therefore getting to McKenzie Pass will take at most 2^{159} \approx 7.308 \cdot 10^{47} hours. But you will get there.

(i) (Bonus.) Use a programmable calculator to redo the last calculation in part (f) in the reverse order:

\[
\frac{1}{1,000,000,000,000} + \frac{1}{999,999,999,999} + \frac{1}{999,999,999,998} + \cdots + \frac{1}{2} + 1.
\]

What do you get?

*Solution:* This seems to be too long a calculation to actually do at current speeds of ordinary computers.