Consider the following sequences:

\[
\begin{align*}
&\left(1\right)_{n=1}^\infty, \quad \left(\sqrt[n]{17}\right)_{n=1}^\infty, \quad \left(n\right)_{n=1}^\infty, \quad \left(n^{17}\right)_{n=1}^\infty, \quad \left(n!\right)_{n=1}^\infty, \\
&\left(1.01^n\right)_{n=1}^\infty, \quad \left(e^n\right)_{n=1}^\infty, \quad \left(200^n\right)_{n=1}^\infty, \quad \left(n^n\right)_{n=1}^\infty, \quad \left(\ln(n)\right)_{n=1}^\infty.
\end{align*}
\]

We want to arrange them in order of how fast they grow, slowest first.

For the purposes of this worksheet, if \((a_n)_{n=1}^\infty\) and \((b_n)_{n=1}^\infty\) are two sequences consisting of strictly positive terms (or at least all but the first several terms are strictly positive), say that “\((b_n)_{n=1}^\infty\) grows much faster than \((a_n)_{n=1}^\infty\)” if \(\lim_{n \to \infty} \frac{a_n}{b_n} = 0\), equivalently, if \(\lim_{n \to \infty} \frac{b_n}{a_n} = \infty\).

1. If \((a_n)_{n=1}^\infty\), \((b_n)_{n=1}^\infty\), and \((c_n)_{n=1}^\infty\) are three sequences consisting of strictly positive terms, such that \((b_n)_{n=1}^\infty\) grows much faster than \((a_n)_{n=1}^\infty\) and \((c_n)_{n=1}^\infty\) grows much faster than \((b_n)_{n=1}^\infty\), what does this say about the relation between \((c_n)_{n=1}^\infty\) and \((a_n)_{n=1}^\infty\)?

2. The result of Part (1) says you don’t need to compare every pair of sequences in the list above. Why?

3. Write down the best guess of your group at the correct order of the sequences in the list above. (Don’t take too long here.)

4. On separate paper, compute enough limits to determine the correct order of the sequences in the list above. Then write these sequences in the appropriate order at the bottom of this page.

Some hints. Most ratios involving \(\ln(n)\) will need to be done by using L’Hopital’s Rule on the corresponding limit of functions. Ratios involving \(n!\) and exponentials \((r^n\) or \(n^n\)) will need to be done by writing out products, as was done in the example in the lecture. Remember that \(r^n = \exp(n \ln(r))\). Also, \(\lim_{n \to \infty} \frac{n^{17}}{e^n} = 0\) can be done using L’Hopital’s Rule directly, but it is much easier to do \(\lim_{n \to \infty} \sqrt[n]{\frac{n^{17}}{e^n}} = 0\) first.

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