For the purposes of this worksheet, if \((a_n)_{n=1}^{\infty}\) and \((b_n)_{n=1}^{\infty}\) are two sequences consisting of strictly positive terms (or at least all but the first several terms are strictly positive), say that “\((b_n)_{n=1}^{\infty}\) grows much faster than \((a_n)_{n=1}^{\infty}\)” if \(\lim_{n \to \infty} \frac{a_n}{b_n} = 0\), equivalently, if \(\lim_{n \to \infty} \frac{b_n}{a_n} = \infty\).

Consider the following sequences:

\[
\begin{align*}
& (1)_{n=1}^{\infty}, \quad (\ln(n))_{n=1}^{\infty}, \quad \left(\sqrt[n]{n}\right)_{n=1}^{\infty}, \quad \left(n\right)_{n=1}^{\infty}, \quad \left(n^{17}\right)_{n=1}^{\infty}, \\
& (1.01^n)_{n=1}^{\infty}, \quad (e^n)_{n=1}^{\infty}, \quad (200^n)_{n=1}^{\infty}, \quad (n!)_{n=1}^{\infty}, \quad (n^n)_{n=1}^{\infty}.
\end{align*}
\]

We want to show that each sequence on this list grows much faster than its predecessor. We have already seen that \((\ln(n))_{n=1}^{\infty}\) grows much faster than \((1)_{n=1}^{\infty}\) and \(\left(\sqrt[n]{n}\right)_{n=1}^{\infty}\) grows much faster than \((\ln(n))_{n=1}^{\infty}\).

1. Show that \((n)_{n=1}^{\infty}\) grows much faster than \(\left(\sqrt[n]{n}\right)_{n=1}^{\infty}\).

2. Show that \(\left(n^{17}\right)_{n=1}^{\infty}\) grows much faster than \((n)_{n=1}^{\infty}\).

3. Show that \((1.01^n)_{n=1}^{\infty}\) grows much faster than \(\left(n^{17}\right)_{n=1}^{\infty}\).

Hint: You can use L’Hopital’s Rule 17 times to show that \(\lim_{x \to \infty} \frac{x^{17}}{e^{\ln(1.01)x}} = 0\), but it is much faster to start by writing \(\frac{x^{17}}{e^{\ln(1.01)x}} = \left(\frac{x}{e^{\ln(1.01)/17x}}\right)^{17}\).

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4. Show that \((e^n)_{n=1}^\infty\) grows much faster than \((1.01^n)_{n=1}^\infty\).

5. Show that \((200^n)_{n=1}^\infty\) grows much faster than \((e^n)_{n=1}^\infty\).

6. Show that \((n!)_{n=1}^\infty\) grows much faster than \((200^n)_{n=1}^\infty\).
   Hint: We did one like this in class Wednesday. Write \(\frac{200^n}{n!} = \left(\frac{200}{1}\right) \left(\frac{200}{2}\right) \cdots \left(\frac{200}{n}\right)\).

7. Show that \((n^n)_{n=1}^\infty\) grows much faster than \((n!)_{n=1}^\infty\).
   Hint: Start in a similar way to the previous problem. First look at the terms for even \(n\).

Bonus problem. Show that \((\ln(n)^2)_{n=1}^\infty\) grows much faster than \((\ln(n))_{n=1}^\infty\) and \((\sqrt[n]{n})_{n=1}^\infty\) grows much faster than \((\ln(n)^2)_{n=1}^\infty\).