MATH 253 (PHILLIPS): SOLUTIONS AND SKETCHES OF SOLUTIONS TO HOMEWORK 2.

This homework sheet is due in class on Wednesday 19 April 2023 (week 3).
All the requirements in the sheet on general instructions for homework apply. In particular, show your work (unlike WeBWorK as done in previous courses), give exact answers (not decimal approximations), and use correct notation. (See the web page on notation.) Also, in particular:

- Use “=” when you are claiming two expressions are equal, but not when you are not making this claim.
- Write “lim_{n→∞}” where it belongs, but not where it doesn’t belong.
- Use enough parentheses.

Example (from the solutions to Homework 1):

\[
\lim_{n→∞} n + 3 = \lim_{n→∞} \frac{1}{n}(n + 3) = \lim_{n→∞} \frac{1 + \frac{3}{n}}{3 + \frac{11}{n}} = \lim_{n→∞} \frac{1 + \lim_{n→∞} \frac{3}{n}}{3 + \lim_{n→∞} \frac{11}{n}} = \frac{1 + 0}{3 + 0} = \frac{1}{3}.
\]

1. (5 points.) A sequence \((a_n)_{n=1}^{∞}\) is given by \((a_n)_{n=1}^{∞} = \left(\frac{1}{5}, \frac{2}{6}, \frac{3}{7}, \frac{4}{8}, \frac{5}{9}, \frac{6}{10}, \frac{7}{11}, \ldots\right)\).

Assuming the pattern continues, write down a formula for \(a_n\) for arbitrary \(n\).

Solution: \(a_n = \frac{(-1)^{n+1}(n+1)}{n+5}\) or \(a_n = \frac{(-1)^{n-1}(n+1)}{n+5}\).

2. (5 points.) The sequence \((a_n)_{n=1}^{∞}\) is given by the formula

\[
a_n = \frac{5^n}{3^{n+2}}
\]

for strictly positive integers \(n\). Determine whether this sequence converges, diverges to \(∞\), diverges to \(-∞\), or diverges in some other way. If it converges, find its limit. (Remember to show your work.)

Solution: We have \(a_n = \left(\frac{5}{3}\right)^n\). Since \(\left|\frac{5}{3}\right| > 1\), we have \(\lim_{n→∞} \left(\frac{5}{3}\right)^n = ∞\). Since \(\frac{5}{3}\) is a strictly positive constant, we also get \(\lim_{n→∞} \left(\frac{1}{9}\right) \left(\frac{5}{3}\right)^n = ∞\).

3. (5 points.) The sequence \((a_n)_{n=1}^{∞}\) is given by the formula

\[
a_n = \frac{\sqrt{n+1}}{9n+1}
\]

for strictly positive integers \(n\). Determine whether this sequence converges, diverges to \(∞\), diverges to \(-∞\), or diverges in some other way. If it converges, find its limit. (Remember to show your work.)

Solution (sketch): We have \(\lim_{n→∞} \frac{n+1}{9n+1} = \frac{1}{9}\) (by standard methods; details omitted), and \(x → \sqrt{x}\) is continuous at \(\frac{1}{9}\), so

\[
\lim_{n→∞} \sqrt{\frac{n+1}{9n+1}} = \sqrt{\frac{1}{9}} = \frac{1}{3}.
\]

Date: 21 April 2023.
4. (5 points.) The sequence \((a_n)_{n=1}^{\infty}\) is given by the formula

\[ a_n = \frac{(-1)^n n^3}{n^3 + 2n^2 + 1} \]

for strictly positive integers \(n\). Determine whether this sequence converges, diverges to \(\infty\), diverges to \(-\infty\), or diverges in some other way. If it converges, find its limit. (Remember to show your work.)

**Solution (sketch):** Standard methods give

\[ \lim_{n \to \infty} a_{2n} = \lim_{n \to \infty} \frac{n^3}{n^3 + 2n^2 + 1} = 1 \quad \text{and} \quad \lim_{n \to \infty} a_{2n+1} = \lim_{n \to \infty} \left( -\frac{n^3}{n^3 + 2n^2 + 1} \right) = -1. \]

The sequence \((a_n)_{n=1}^{\infty}\) can’t converge to two different limits, so this sequence diverges.

5. (5 points.) The sequence \((a_n)_{n=1}^{\infty}\) is given by the formula

\[ a_n = \cos\left(\frac{2}{n}\right) \]

for strictly positive integers \(n\). Determine whether this sequence converges, diverges to \(\infty\), diverges to \(-\infty\), or diverges in some other way. If it converges, find its limit. (Remember to show your work.)

**Solution:** We have \(\lim_{n \to \infty} \frac{2}{n} = 0\), and \(x \mapsto \cos(x)\) is continuous at 0, so \(\lim_{n \to \infty} \cos(2/n) = \cos(0) = 1\).

6. (5 points.) The sequence \((a_n)_{n=1}^{\infty}\) is given by the formula

\[ a_n = \sqrt[1+3n]{2} \]

for strictly positive integers \(n\). Determine whether this sequence converges, diverges to \(\infty\), diverges to \(-\infty\), or diverges in some other way. If it converges, find its limit. (Remember to show your work.)

**Solution (sketch):** We have

\[ \ln(a_n) = \frac{1}{n} \ln(2^{1+3n}) = \left( \frac{1}{n} \right) (1 + 3n) \ln(2) = \frac{(1 + 3n) \ln(2)}{n}. \]

So, by standard methods, \(\lim_{n \to \infty} \ln(a_n) = 3 \ln(2)\). Since \(x \mapsto e^x\) is continuous at \(3 \ln(2)\), it follows that

\[ \lim_{n \to \infty} = \lim_{n \to \infty} e^{\ln(a_n)} = e^{3 \ln(2)} = 2^3 = 8. \]

7. (5 points.) The sequence \((a_n)_{n=1}^{\infty}\) is given by the formula

\[ a_n = \frac{(2n-1)!}{(2n+1)!} \]

for strictly positive integers \(n\). Determine whether this sequence converges, diverges to \(\infty\), diverges to \(-\infty\), or diverges in some other way. If it converges, find its limit. (Remember to show your work.)

**Solution:** Rewrite

\[ \frac{(2n-1)!}{(2n+1)!} = \frac{1}{2n(2n+1)}. \]

Now the limit is clearly 0.

8. (5 points.) The sequence \((a_n)_{n=1}^{\infty}\) is given by the formula

\[ a_n = \frac{\sin(2n)}{1 + \sqrt{n}} \]

for strictly positive integers \(n\). Determine whether this sequence converges, diverges to \(\infty\), diverges to \(-\infty\), or diverges in some other way. If it converges, find its limit. (Remember to show your work.)
Solution: We have
\[-\frac{1}{1 + \sqrt{n}} \leq \frac{\sin(2n)}{1 + \sqrt{n}} \leq \frac{1}{1 + \sqrt{n}}\]
for all strictly positive integers \(n\), and
\[
\lim_{n \to \infty} \left( -\frac{1}{1 + \sqrt{n}} \right) = \lim_{n \to \infty} \frac{1}{1 + \sqrt{n}} = 0,
\]
so
\[
\lim_{n \to \infty} \frac{\sin(2n)}{1 + \sqrt{n}} = 0
\]
by the Squeeze Theorem for sequences.

9. (5 points.) The sequence \((a_n)_{n=1}^\infty\) is given by the formula
\[a_n = \frac{n^3 + \sin(n)}{13n^3 + 3 - \cos(n)}\]
for strictly positive integers \(n\). Determine whether this sequence converges, diverges to \(\infty\), diverges to \(-\infty\), or diverges in some other way. If it converges, find its limit. (Remember to show your work.)

Solution: We have
\[-\frac{1}{n^3} \leq \frac{\sin(n)}{n^3} \leq \frac{1}{n^3}\]
and
\[-\frac{1}{n^3} \leq \frac{\cos(n)}{n^3} \leq \frac{1}{n^3}\]
for all strictly positive integers \(n\). Therefore the Squeeze Theorem implies
\[
\lim_{n \to \infty} \frac{\sin(n)}{n^3} = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{\cos(n)}{n^3} = 0.
\]
Now
\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n^3 + \sin(n)}{13n^3 + 3 - \cos(n)} = \lim_{n \to \infty} \frac{\frac{1}{n^3}(n^3 + \sin(n))}{\frac{1}{13}[13n^3 + 3 - \cos(n)]} = \lim_{n \to \infty} \frac{1 + \frac{\sin(n)}{n^3}}{13 + \frac{3}{n} - \frac{\cos(n)}{n}} = \frac{1}{13} + 0 + 0 = \frac{1}{13}.
\]

Attempted solutions using L’Hospital’s Rule on the function
\[f(x) = \frac{x^3 + \sin(x)}{13x^3 + 3 - \cos(n)}\]
will not work, since \(\lim_{x \to \infty} f(x)\) can’t be found using L’Hospital’s Rule.

10. (5 points.) The sequence \((a_n)_{n=1}^\infty\) is given by the formula
\[a_n = \frac{n!}{3^n}\]
for strictly positive integers \(n\). Determine whether this sequence converges, diverges to \(\infty\), diverges to \(-\infty\), or diverges in some other way. If it converges, find its limit. (Remember to show your work.)

Solution: For every positive integer \(n\), we have
\[a_n = \frac{n!}{3^n} = \left( \frac{1}{3} \right) \left( \frac{2}{3} \right) \left( \frac{3}{3} \right) \left( \frac{4}{3} \right) \left( \frac{5}{3} \right) \cdots \left( \frac{n}{3} \right)\]
\[\geq \left( \frac{1}{3} \right) \left( \frac{2}{3} \right) \left( \frac{3}{3} \right) \left( \frac{4}{3} \right) \left( \frac{4}{3} \right) \cdots \left( \frac{4}{3} \right)\]
\[= \left( \frac{1}{3} \right) \left( \frac{2}{3} \right) \left( \frac{3}{3} \right) \cdots \left( \frac{4}{3} \right)^{n-3} = 2 \left( \frac{4}{9} \right)^{n-3}.
\]
That is,
\[a_n \geq \frac{2}{9} \left( \frac{4}{3} \right)^{n-3}.
\]
Since $\frac{3}{4} > 1$, we have
\[
\lim_{n \to \infty} \left( \frac{4}{3} \right)^n = \infty,
\]
so
\[
\lim_{n \to \infty} \frac{2}{9} \left( \frac{4}{3} \right)^{n-3} = \frac{2}{9} \lim_{n \to \infty} \left( \frac{4}{3} \right)^{n-3} = \infty.
\]
Therefore also
\[
\lim_{n \to \infty} \frac{n!}{3^n} = \infty.
\]

11. (5 points.) The sequence $\left( a_n \right)_{n=1}^{\infty}$ is given by the formula
\[a_n = n(-1)^n\]
for strictly positive integers $n$. Determine whether this sequence is strictly increasing, strictly decreasing, or not monotonic. Also determine whether it is bounded.

Solution: This sequence is not strictly increasing, since $a_3 = -3 < a_2 = 2$. It is not strictly decreasing, since $a_2 = 2 > a_1 = -1$. So it is not monotonic. This sequence is also clearly not bounded.

12. (5 points.) The sequence $\left( a_n \right)_{n=1}^{\infty}$ is given by the formula
\[a_n = n + \frac{1}{n}\]
for strictly positive integers $n$. Determine whether this sequence is strictly increasing, strictly decreasing, or not monotonic. Also determine whether it is bounded.

Solution: We show that $\left( a_n \right)_{n=1}^{\infty}$ is strictly increasing. Let $n$ be a strictly positive integer. Then
\[
a_{n+1} - a_n = n + 1 + \frac{1}{n+1} - n - \frac{1}{n} = 1 + \frac{1}{n+1} - \frac{1}{n} = \frac{n(n+1) + n - (n+1)}{n(n+1)} = \frac{n^2 + n - 1}{n(n+1)}.
\]
Since $n \geq 1$, we have $n^2 + n - 1 > 0$, so $a_{n+1} - a_n > 0$.

13. (5 points.) The sequence $\left( a_n \right)_{n=1}^{\infty}$ is given by the formula
\[a_n = \frac{2n+1}{n+1}\]
for strictly positive integers $n$. Determine whether this sequence is strictly increasing, strictly decreasing, or not monotonic. Also determine whether it is bounded.

Solution: It is easier to look at
\[a_n - 2 = \frac{2n+1}{n+1} - 2 = -\frac{1}{n+1}.
\]
Since the sequence $\left( n+1 \right)_{n=1}^{\infty}$ is strictly positive and strictly increasing, the sequence $\left( \frac{1}{n+1} \right)_{n=1}^{\infty}$ is strictly decreasing, so $\left( -\frac{1}{n+1} \right)_{n=1}^{\infty}$ is strictly increasing, so
\[\left( a_n \right)_{n=1}^{\infty} = \left( \frac{2 - \frac{1}{n+1}}{n+1} \right)_{n=1}^{\infty}
\]
is strictly increasing.
All convergent sequences are bounded, and $\lim_{n \to \infty} a_n = 2$, so $\left( a_n \right)_{n=1}^{\infty}$ is bounded.
Alternate solution for monontonicity: We calculate, for every strictly positive integer \( n \):
\[
\frac{a_n - a_{n+1}}{n+1} = \frac{2n+1 - 2(n+1)+1}{n+1} = \frac{2n+1 - 2n+3}{n+1} = \frac{2n+3}{n+1}
\]
\[
\frac{(2n+1)(n+2) - (2n+3)(n+1)}{(n+1)(n+2)} = \frac{(2n^2+5n+2) - (2n^2+5n+3)}{(n+1)(n+2)} = -\frac{1}{(n+1)(n+2)} < 0.
\]

Therefore \( a_{n+1} < a_n \). This shows that \((a_n)_{n=1}^\infty\) is strictly increasing.

14. (5 points.) Let \((b_n)_{n=1}^\infty\) be a sequence with \( b_n > 0 \) for every strictly positive integer \( n \). Suppose \( \lim_{n \to \infty} \ln(b_n) = -\infty \). What, if anything, does that say about \( \lim_{n \to \infty} b_n \)?

**Solution:** Set \( a_n = \ln(b_n) \). Thus \( \lim_{n \to \infty} a_n = -\infty \).

We have \( \lim_{x \to -\infty} e^x = 0 \). Therefore \( \lim_{n \to \infty} e^{an} = 0 \). Since \( e^{an} = b_n \), this says that \( \lim_{n \to \infty} b_n = 0 \).

15. (5 points.) Consider the sequence given by \( a_n = \sqrt[3]{n} \) for strictly positive integers \( n \). You want to prove that \( \lim_{n \to \infty} a_n = \infty \). So, for example, for \( M = 19 \) you should be able to find some integer \( N > 0 \) such that for all \( n > N \) we have \( a_n > M \). Find such a value of \( N \), and show that it works. (You need not find the best value of \( N \).) (In more colloquial wording, how many terms are needed for the sequence to get and stay bigger than \( 19 \)?)

**Solution:** I am going to choose \( N = 8000 = 20^3 \). (This is not the best choice.) Consider an arbitrary integer \( n \) such that \( n > 8000 \). Then \( a_n = \sqrt[3]{n} > \sqrt[3]{8000} = 20 \). In particular, \( a_n > 19 \). So \( N = 8000 \) works.

(For reference, the smallest choice which works is \( N = 19^3 = 6859 \).)

16. (5 points/part.) Consider the sequence given by \( a_n = \frac{n^2 - 2}{n^2} \) for strictly positive integers \( n \).

(a) Find the limit \( L \) of this sequence.

**Solution:** Rewrite \( a_n = 1 - \frac{2}{n^2} \). Since \( \lim_{n \to \infty} \frac{2}{n^2} = 0 \), we have \( \lim_{n \to \infty} a_n = 1 \).

(b) Find an integer \( N \) that has the property that \( |a_n - L| < 0.01 \) for every \( n \geq N \). You need not find the best value of \( N \).

**Solution:** Start with the fact that we want \( |a_n - 1| < 0.01 \). Rewrite this as
\[
\left| 1 - \frac{2}{n^2} \right| < 0.01
\]
which says
\[
\frac{2}{n^2} < 0.01.
\]
Rearrange to get
\[
n^2 > \frac{2}{0.01} = 200
\]
and so \( n > \sqrt{200} \approx 14.14 \). So for \( n \geq 15 \) we will have \( |a_n - 1| < 0.01 \).

17. (5 points/part.) Consider the sequence given by \( a_n = 13 \cdot (0.92)^n \) for strictly positive integers \( n \).

(a) Find the limit \( L \) of this sequence.

**Solution:** This is a geometric sequence. Since \( 0 < 0.92 < 1 \), the sequence converges to 0.

(b) Find an integer \( N \) that has the property that \( |a_n - L| < 0.01 \) for every \( n \geq N \). (In more colloquial wording, how many terms are needed for the sequence to get within 0.01 of the limit and stay there?) You need not find the best value of \( N \).
Solution: This is equivalent to
\[(0.92)^n < \frac{0.01}{13}\]
and taking natural logs gives
\[n \ln(0.92) < \ln\left(\frac{0.01}{13}\right) = -\ln(1300).\]
Since \(0.92 < 1\), \(\ln(0.92) < 0\). Dividing therefore changes the direction of the inequality, and we get
\[n > \frac{-\ln(1300)}{\ln(0.92)} \approx 85.99.\]
So we need \(n \geq 86\) to have \(|a_n - 0| < 0.01\).

(c) Find an integer \(N\) that has the property that \(|a_n - L| < 0.001\) for every \(n \geq N\). You need not find the best value of \(N\).

Solution: By the same analysis, this time we need
\[n > \frac{-\ln(13000)}{\ln(0.92)} \approx 113.6.\]
So we need \(n \geq 114\) to have \(|a_n - 0| < 0.001\).

Extra credit. Prove directly from the definition that
\[\lim_{n \to \infty} \frac{6n + 9}{2n} = 3.\]
(This is like several of the earlier problems, except that those problems asked for choices of \(N\) to go with specific values of \(\varepsilon\), such as \(\varepsilon = 0.01\), but here you need to find a choice of \(N\) for an arbitrary \(\varepsilon > 0\). Your choice of \(N\) will, of course, need to depend on \(\varepsilon\).)

Solution: Let \(\varepsilon > 0\) be arbitrary. Choose an integer \(N\) such that \(N > 9/(2\varepsilon)\). Let \(n\) be an arbitrary integer satisfying \(n > N\). Then
\[\left|\frac{6n + 9}{2n} - 3\right| = \left|\frac{9}{2n}\right| = \frac{9}{2n} < \frac{9}{2N} < \varepsilon.\]
This completes the proof.