SOLUTIONS TO WORKSHEET: MORE SERIES YOU CAN SUM; INTEGRAL TEST

Names and student IDs: Solutions [πππ−ππ−ππππ]

1. Determine whether the series \( \sum_{n=0}^{\infty} \frac{(-2)^n}{7^n} \) is convergent or divergent. If it is convergent, find the sum.

*Solution:* This series is the geometric series \( \sum_{n=0}^{\infty} r^n \) with \( r = \frac{-2}{7} \). Since \( \left| \frac{-2}{7} \right| < 1 \), the series converges, and the sum is
\[
\sum_{n=0}^{\infty} \frac{(-2)^n}{7^n} = \frac{1}{1 - \left( \frac{-2}{7} \right)} = \frac{7}{9}.
\]

2. Determine whether the series \( \sum_{n=3}^{\infty} \frac{(-2)^n}{7^n+5} \) is convergent or divergent. If it is convergent, find the sum.

(Why does convergence have the same answer as in Problem 1? If the series in Problem 1 converges, what factor do you need to multiply its sum by to get the sum of this one?)

*Solution:* The series is a geometric series with the same common ratio as the one in Problem 1, namely \( \frac{-2}{7} \), but the starting point is different. Probably the easiest way to find the sum is to write it out as:
\[
\frac{(-2)^3}{7^8} + \frac{(-2)^4}{7^9} + \frac{(-2)^5}{7^{10}} + \cdots = \frac{(-2)^3}{7^8} \left( 1 + \frac{-2}{7} + \left( \frac{-2}{7} \right)^2 + \cdots \right)
= \frac{(-2)^3}{7^8} \sum_{n=0}^{\infty} \left( \frac{-2}{7} \right)^n = \frac{(-2)^3}{7^8} \left( \frac{7}{9} \right) = - \frac{8}{9 \cdot 7^7}.
\]

3. Consider the two telescoping series
\[
\sum_{n=1}^{\infty} \left[ \sqrt{n} - \sqrt{n+1} \right] \quad \text{and} \quad \sum_{n=1}^{\infty} \left( \sqrt{\frac{17}{n}} - \sqrt{\frac{17}{n+1}} \right).
\]
One of these converges, and one diverges. Which is which, and why?

*Solution:* The \( n \)th partial sum \( s_n \) of the first series is
\[
s_n = \sum_{k=1}^{n} \left[ \sqrt{k} - \sqrt{k+1} \right] = \sqrt{1} - \sqrt{n+1}.
\]

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Since \( \lim_{n \to \infty} \sqrt{n} = \infty \), it follows that
\[
\sum_{n=1}^{\infty} [\sqrt{n} - \sqrt{n+1}] = \lim_{n \to \infty} s_n = \lim_{n \to \infty} [\sqrt{1} - \sqrt{n+1}] = -\infty.
\]

Thus, the series diverges.

The \( n \)th partial sum \( t_n \) of the second series is
\[
t_n = \sum_{k=1}^{n} \left( \sqrt{3 + \frac{17}{1}} - \sqrt{3 + \frac{17}{k+1}} \right) = \sqrt{3 + \frac{17}{1}} - \sqrt{3 + \frac{17}{n+1}}.
\]

Since \( \lim_{n \to \infty} \frac{17}{n} = 0 \), we have
\[
\lim_{n \to \infty} \sqrt{3 + \frac{17}{n+1}} = \sqrt{3}.
\]

Therefore
\[
\sum_{n=1}^{\infty} [\sqrt{n} - \sqrt{n+1}] = \lim_{n \to \infty} t_n = \lim_{n \to \infty} \left( \sqrt{3 + \frac{17}{1}} - \sqrt{3 + \frac{17}{n+1}} \right) = \sqrt{20} - \sqrt{3}.
\]
Thus, this series converges.

4. Use the Integral Test to show that \( \sum_{n=1}^{\infty} \frac{1}{n^\pi} \) converges. **Be sure to check that all the hypotheses of the Integral Test actually hold!**

**Homework and exam problems in which this is not done will lose substantial credit.**

**Solution:** Define \( f(x) = x^{-\pi} \) for \( x > 0 \). This function is nonnegative and nonincreasing (probably “decreasing” in the textbook: it is actually strictly decreasing). The series has the form \( \sum_{n=1}^{\infty} a_n \) with \( a_n = f(n) \) for \( n = 1, 2, \ldots \). Therefore the all the hypotheses of the Integral Test actually hold.

Now
\[
\int_{1}^{\infty} x^{-\pi} \, dx = \lim_{b \to \infty} \int_{1}^{b} x^{-\pi} \, dx = \lim_{b \to \infty} \left( \frac{x^{-\pi+1}}{-\pi+1} \right)_{1}^{b} = \lim_{b \to \infty} \left( \frac{b^{-\pi+1}}{-\pi+1} - \frac{1^{-\pi+1}}{-\pi+1} \right).
\]

Since \(-\pi + 1 < 0\), we have \( \lim_{b \to \infty} b^{-\pi+1} = 0 \). Therefore
\[
\int_{1}^{\infty} x^{-\pi} \, dx = -\frac{1^{-\pi+1}}{-\pi}.
\]

In particular, this improper integral converges. Therefore, by the Integral Test, \( \sum_{n=1}^{\infty} \frac{1}{n^\pi} \) converges (but to something else.)