SOLUTIONS TO WORKSHEET: INTEGRAL TEST FOR CONVERGENCE

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1. Use the Integral Test to determine whether the series \( \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \) converges. You will use a suitable function \( f \); make sure to check that your choice of \( f \) satisfies the two conditions it is supposed to.

Solution: We take \( f(x) = \frac{1}{x^2 + 1} \). We need to check that \( f(x) \geq 0 \) beyond some point; this is clearly true for all real \( x \). We need to check that \( f \) is nonincreasing beyond some point; since \( x \mapsto x^2 + 1 > 0 \) and is increasing on \([0, \infty)\), clearly \( x \mapsto \frac{1}{x^2 + 1} \) is decreasing on \([0, \infty)\).

Now \( \int_{0}^{\infty} \frac{1}{x^2 + 1} \, dx = \arctan(x) \bigg|_{0}^{\infty} = \lim_{x \to \infty} \arctan(x) - \arctan(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2} \).

So the improper integral converges. Therefore \( \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \) converges.

2. Use the Integral Test to determine whether the series \( \sum_{n=1}^{\infty} \frac{[\ln(n)]^2}{n} \) converges. You will use a suitable function \( f \); make sure to check that your choice of \( f \) satisfies the two conditions it is supposed to.

Hint: One way to check that a function is decreasing is to look at its derivative.

Solution: We take \( f(x) = \frac{[\ln(x)]^2}{x} \). We need to check that \( f(x) \geq 0 \) beyond some point; this is clearly true for all \( x > 0 \). We also need to check that \( f \) is nonincreasing beyond some point. For this, we calculate \( f'(x) = \frac{[2 \ln(x) \cdot \frac{1}{x} - [\ln(x)]^2}{x^2} = \frac{2 \ln(x) - [\ln(x)]^2}{x^2} = \frac{[2 - \ln(x)]\ln(x)}{x^2} \).

If \( x > e^2 \) then \( \ln(x) > 2 \), so \( f'(x) < 0 \). So \( f \) is decreasing on \((e^2, \infty)\).

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Using the substitution \( u = \ln(x) \), so \( du = \frac{1}{x} \, dx \), we get
\[
\int_1^\infty \frac{[\ln(x)]^2}{x} \, dx = \left( \frac{1}{3} [\ln(x)]^3 \right) \bigg|_1^\infty.
\]
Since \( \lim_{x \to \infty} \frac{1}{3} [\ln(x)]^3 = \infty \), the improper integral diverges. Therefore
\[
\sum_{n=1}^{\infty} \frac{[\ln(n)]^2}{n}
\]
diverges.

3. Consider the two telescoping series
\[
\sum_{n=1}^{\infty} \left( \sqrt[3]{n} - \sqrt[3]{n+1} \right) \quad \text{and} \quad \sum_{n=1}^{\infty} \left( \sqrt{11 + \frac{219}{n}} - \sqrt{11 + \frac{219}{n+1}} \right).
\]
One of these converges, and one diverges. Which is which, and why?

**Solution:** The \( n \)th partial sum \( s_n \) of the first series is
\[
s_n = \sum_{k=1}^{n} \left( \sqrt[3]{k} - \sqrt[3]{k+1} \right) = \sqrt[3]{1} - \sqrt[3]{n+1}.
\]
Since \( \lim_{n \to \infty} \sqrt[3]{n} = \infty \), it follows that
\[
\sum_{n=1}^{\infty} \left( \sqrt[3]{n} - \sqrt[3]{n+1} \right) = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \left( \sqrt[3]{1} - \sqrt[3]{n+1} \right) = -\infty.
\]
Thus, the series diverges.

The \( n \)th partial sum \( t_n \) of the second series is
\[
t_n = \sum_{k=1}^{n} \left( \sqrt{11 + \frac{219}{k}} - \sqrt{11 + \frac{219}{k+1}} \right) = \sqrt{11 + \frac{219}{1}} - \sqrt{11 + \frac{219}{n+1}}.
\]
Since \( \lim_{n \to \infty} \frac{219}{n+1} = 0 \), we have
\[
\lim_{n \to \infty} \sqrt{11 + \frac{219}{n+1}} = \sqrt{11}.
\]
Therefore
\[
\sum_{n=1}^{\infty} \left( \sqrt{11 + \frac{219}{n}} - \sqrt{11 + \frac{219}{n+1}} \right) = \lim_{n \to \infty} t_n
\]
\[
= \lim_{n \to \infty} \left( \sqrt{11 + \frac{219}{1}} - \sqrt{11 + \frac{219}{n+1}} \right)
\]
\[
= \sqrt{230} - \sqrt{11}.
\]
Thus, this series converges.