SOLUTIONS TO WORKSHEET: INTEGRAL TEST FOR CONVERGENCE 2

Names and student IDs: Solutions ππππ-ππππ-πππππ

1. Determine whether the series \( \sum_{n=1}^{\infty} \frac{1}{n^{2/3}} \) converges or diverges.

(Using a theorem from Tuesday, you should be able to give a two sentence reason without actually carrying out any tests.)

Solution: The series has the form \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) with \( p = \frac{2}{3} \). Since \( \frac{2}{3} < 1 \), the series diverges.

2. Determine whether the series \( \sum_{n=13}^{\infty} \frac{178}{n^{7/3}} \) converges or diverges. Write your reasoning in a mathematically and notationally correct form.

(This one is nearly as easy as the first problem.)

Solution: The series \( \sum_{n=1}^{\infty} \frac{1}{n^{7/3}} \) has the form \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) with \( p = \frac{7}{3} \). Since \( \frac{7}{3} > 1 \), this series diverges.

Therefore also the series \( \sum_{n=13}^{\infty} \frac{1}{n^{7/3}} \) converges (although not to the same sum. Now
\[
\sum_{n=13}^{\infty} \frac{178}{n^{7/3}} = 178 \sum_{n=13}^{\infty} \frac{1}{n^{7/3}}
\]
converges.

3. Use the Integral Test to determine whether the series \( \sum_{n=1}^{\infty} \frac{1}{16n^2 + 1} \) converges. You will use a suitable function \( f \); make sure to check that your choice of \( f \) satisfies the two conditions it is supposed to. (You should not need methods from Math 251 to check the hypotheses: they follow quickly from ordinary algebra.)

Date: 26 April 2023.
Solution: We take \( f(x) = \frac{1}{16x^2 + 1} \). We need to check that \( f(x) \geq 0 \) beyond some point; this is clearly true for all real \( x \). We need to check that \( f \) is nonincreasing beyond some point; since \( x \mapsto \frac{1}{x^2 + 1} > 0 \) and is increasing on \([0, \infty)\), clearly \( x \mapsto \frac{1}{x^2 + 1} \) is decreasing on \([0, \infty)\).

Now, using the substitution \( u = 4x \), so \( du = \frac{1}{4} \, dx \),

\[
\int_{0}^\infty \frac{1}{16x^2 + 1} \, dx = \frac{1}{4} \arctan(u) + C = \frac{1}{4} \arctan(4x) + C.
\]
Therefore

\[
\int_{0}^\infty \frac{1}{16x^2 + 1} \, dx = \frac{1}{4} \arctan(4x) \bigg|_{0}^{\infty} = \lim_{b \to \infty} \left( \frac{1}{4} \arctan(4x) \bigg|_{0}^{b} \right) = \frac{1}{4} \lim_{b \to \infty} \left( \arctan(4b) - \frac{1}{4} \arctan(0) \right) = \frac{\pi}{8} - 0 = \frac{\pi}{8}.
\]
So the improper integral converges. Therefore \( \sum_{n=1}^{\infty} \frac{1}{16n^2 + 1} \) converges.

4. Use the Integral Test to determine whether the series \( \sum_{n=1}^{\infty} \frac{[\ln(n)]^3}{n} \) converges. You will use a suitable function \( f \); make sure to check that your choice of \( f \) satisfies the two conditions it is supposed to.

(You will need methods from Math 251 to check the hypotheses of the integral test.)

Solution: We take \( f(x) = \frac{[\ln(x)]^3}{x} \). We need to check that \( f(x) \geq 0 \) beyond some point; this is clearly true for all \( x > 1 \). We also need to check that \( f \) is nonincreasing beyond some point. For this, we calculate

\[
f'(x) = \frac{3[\ln(x)]^2 \cdot \left( \frac{1}{x} \right) \cdot x - [\ln(x)]^3}{x^2} = \frac{3[\ln(x)]^2 - [\ln(x)]^3}{x^2} = \frac{[3 \ln(x)] \ln(x)}{x^2}.
\]
If \( x > e^3 \) then \( \ln(x) > 3 \), so \( f'(x) < 0 \). So \( f \) is decreasing on \((e^3, \infty)\).

Using the substitution \( u = \ln(x) \), so \( du = \frac{1}{x} \, dx \), we get

\[
\int_{1}^{\infty} \frac{[\ln(x)]^3}{x} \, dx = \left( \frac{1}{4} [\ln(x)]^4 \right) \bigg|_{1}^{\infty}.
\]
Since \( \lim_{x \to \infty} \frac{1}{4} [\ln(x)]^4 = \infty \), the improper integral diverges. Therefore \( \sum_{n=1}^{\infty} \frac{[\ln(n)]^3}{n} \) diverges.
Do not start the integral at 0. The integral \( \int_0^\infty \frac{[\ln(x)]^3}{x} \, dx \) diverges because of bad behavior at 0, which hides the part relevant to the integral test.

5. Determine whether the series \( \sum_{n=1}^{\infty} \left( \sqrt[7]{11 + \frac{781}{n}} - \sqrt[7]{11 + \frac{781}{n+1}} \right) \) converges or diverges. If it converges, what is the sum? Write your reasoning in a mathematically and notationally correct form.

**Solution:** The \( n \)th partial sum \( s_n \) of the series is
\[
s_n = \sum_{k=1}^{n} \left( \sqrt[7]{11 + \frac{781}{k}} - \sqrt[7]{11 + \frac{781}{k+1}} \right) = \sqrt[7]{11 + \frac{781}{1}} - \sqrt[7]{11 + \frac{781}{n+1}}.\]

Since \( \lim_{n \to \infty} \frac{781}{n+1} = 0 \), we have
\[
\lim_{n \to \infty} \sqrt[7]{11 + \frac{781}{n+1}} = \sqrt[7]{11}.
\]

Therefore
\[
\sum_{n=1}^{\infty} \left( \sqrt[7]{11 + \frac{781}{n}} - \sqrt[7]{11 + \frac{781}{n+1}} \right) = \lim_{n \to \infty} s_n
\]
\[
= \lim_{n \to \infty} \left( \sqrt[7]{11 + \frac{781}{1}} - \sqrt[7]{11 + \frac{781}{n+1}} \right)
\]
\[
= \sqrt[7]{791} - \sqrt[7]{11}.
\]
Thus, this series converges to \( \sqrt[7]{791} - \sqrt[7]{11} \).

6. Determine whether the series \( \sum_{n=13}^{\infty} \left( \frac{178}{n^{7/3}} + \frac{1}{4^n} \right) \) converges or diverges. Write your reasoning in a mathematically and notationally correct form.

**Solution:** We saw in Problem 2 above that \( \sum_{n=13}^{\infty} \frac{178}{n^{7/3}} \) converges. The series \( \sum_{n=13}^{\infty} \frac{1}{4^n} \) is a geometric series with common ration \( \frac{1}{4} \), and \( -1 < \frac{1}{4} < 1 \), so the series converges. Therefore \( \sum_{n=13}^{\infty} \left( \frac{178}{n^{7/3}} + \frac{1}{4^n} \right) \) converges.