SOLUTIONS TO WORKSHEET: INTEGRAL TEST FOR CONVERGENCE 2

Names and student IDs: Solutions $\pi\pi\pi-\pi\pi\pi\pi\pi$

1. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ converges or diverges.

(Using a theorem from Tuesday, you should be able to give a two sentence reason without actually carrying out any tests.)

Solution: The series has the form $\sum_{n=1}^{\infty} \frac{1}{n^{p}}$ with $p = \frac{2}{3}$. Since $\frac{2}{3} < 1$, the series diverges.

2. Determine whether the series $\sum_{n=13}^{\infty} \frac{178}{n^{7/3}}$ converges or diverges. Write your reasoning in a mathematically and notationally correct form.

(This one is nearly as easy as the first problem.)

Solution: The series $\sum_{n=1}^{\infty} \frac{1}{n^{7/3}}$ has the form $\sum_{n=1}^{\infty} \frac{1}{n^{p}}$ with $p = \frac{7}{3}$. Since $\frac{7}{3} > 1$, this series diverges.

Therefore also the series $\sum_{n=13}^{\infty} \frac{1}{n^{7/3}}$ converges (although not to the same sum. Now

$$\sum_{n=13}^{\infty} \frac{178}{n^{7/3}} = 178 \sum_{n=13}^{\infty} \frac{1}{n^{7/3}}$$

converges.

3. Use the Integral Test to determine whether the series $\sum_{n=1}^{\infty} \frac{1}{16n^2 + 1}$ converges. You will use a suitable function $f$; make sure to check that your choice of $f$ satisfies the two conditions it is supposed to. (You should not need methods from Math 251 to check the hypotheses: they follow quickly from ordinary algebra.)

Date: 26 April 2023.
Solution: We take \( f(x) = \frac{1}{16x^2 + 1} \). We need to check that \( f(x) \geq 0 \) beyond some point; this is clearly true for all real \( x \). We need to check that \( f \) is nonincreasing beyond some point; since \( x \mapsto 16x^2 + 1 > 0 \) and is increasing on \([0, \infty)\), clearly \( x \mapsto \frac{1}{x^2 + 1} \) is decreasing on \([0, \infty)\).

Now, using the substitution \( u = 4x \), so \( du = \frac{1}{4} dx \),

\[
\int_0^\infty \frac{1}{16x^2 + 1} dx = \frac{1}{4} \arctan(4x) + C = \frac{1}{4} \arctan(4x) + C.
\]

Therefore

\[
\int_0^\infty \frac{1}{16x^2 + 1} dx = \frac{1}{4} \arctan(4b) - \frac{1}{4} \arctan(0) = \frac{\pi}{8} - 0 = \frac{\pi}{8}.
\]

So the improper integral converges. Therefore \( \sum_{n=1}^{\infty} \frac{1}{16n^2 + 1} \) converges.

4. Use the Integral Test to determine whether the series \( \sum_{n=1}^{\infty} \frac{[\ln(n)]^3}{n} \) converges. You will use a suitable function \( f \); make sure to check that your choice of \( f \) satisfies the two conditions it is supposed to.

(You will need methods from Math 251 to check the hypotheses of the integral test.)

Solution: We take \( f(x) = \frac{[\ln(x)]^3}{x} \). We need to check that \( f(x) \geq 0 \) beyond some point; this is clearly true for all \( x > 0 \). We also need to check that \( f \) is nonincreasing beyond some point. For this, we calculate

\[
f'(x) = \frac{[3\ln(x)]^2 \left( \frac{1}{x} \right) - [\ln(x)]^3}{x^2} = \frac{3[\ln(x)]^2 - [\ln(x)]^3}{x^2} = \frac{[3 - \ln(x)][\ln(x)]^2}{x^2}.
\]

If \( x > e^3 \) then \( \ln(x) > 3 \), so \( f'(x) < 0 \). So \( f \) is decreasing on \((e^3, \infty)\).

Using the substitution \( u = \ln(x) \), so \( du = \frac{1}{x} dx \), we get

\[
\int_1^\infty \frac{[\ln(x)]^3}{x} dx = \left( \frac{1}{4}[\ln(u)]^4 \right)_1^\infty.
\]

Since \( \lim_{u \to \infty} \frac{1}{4}[\ln(u)]^4 = \infty \), the improper integral diverges. Therefore \( \sum_{n=1}^{\infty} \frac{[\ln(n)]^3}{n} \) diverges.
Do not start the integral at 0. The integral \( \int_0^\infty \frac{[\ln(x)]^3}{x} \, dx \) diverges because of bad behavior at 0, which hides the part relevant to the integral test.

5. Determine whether the series \( \sum_{n=1}^{\infty} \left( \sqrt[7]{11 + \frac{781}{n}} - \sqrt[7]{11 + \frac{781}{n+1}} \right) \) converges or diverges. If it converges, what is the sum? Write your reasoning in a mathematically and notationally correct form.

\textbf{Solution:} The \( n \)th partial sum \( s_n \) of the series is

\[ s_n = \sum_{k=1}^{n} \left( \sqrt[7]{11 + \frac{781}{k}} - \sqrt[7]{11 + \frac{781}{k+1}} \right) = \sqrt[7]{11 + \frac{781}{1}} - \sqrt[7]{11 + \frac{781}{n+1}}. \]

Since \( \lim_{n \to \infty} \frac{781}{n+1} = 0 \), we have

\[ \lim_{n \to \infty} \sqrt[7]{11 + \frac{781}{n+1}} = \sqrt[7]{11}. \]

Therefore

\[ \sum_{n=1}^{\infty} \left( \sqrt[7]{11 + \frac{781}{n}} - \sqrt[7]{11 + \frac{781}{n+1}} \right) = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \left( \sqrt[7]{11 + \frac{781}{1}} - \sqrt[7]{11 + \frac{781}{n+1}} \right) = \sqrt[7]{791} - \sqrt[7]{11}. \]

Thus, this series converges to \( \sqrt[7]{791} - \sqrt[7]{11} \).

6. Determine whether the series \( \sum_{n=13}^{\infty} \frac{178}{n^{7/3}} + \frac{3}{4^n} \sum_{n=13}^{\infty} \frac{178}{n^{7/3}} + \frac{1}{4^n} \) converges or diverges. Write your reasoning in a mathematically and notionally correct form.

\textbf{Solution:} We saw in Problem 2 above that \( \sum_{n=13}^{\infty} \frac{178}{n^{7/3}} \) converges. The series \( \sum_{n=13}^{\infty} \frac{1}{4^n} \) is a geometric series with common ratio \( \frac{1}{4} \), and \(-1 < \frac{1}{4} < 1\), so the series converges. Therefore \( \sum_{n=13}^{\infty} \frac{178}{n^{7/3}} + \frac{3}{4^n} \sum_{n=13}^{\infty} \frac{178}{n^{7/3}} + \frac{1}{4^n} \) converges.