WORKSHEET: COMPARISON TEST

Names and student IDs: Solutions [πππ-ππππππ]

Comparison test for convergence of series:

(1) If $0 \leq a_n \leq b_n$ for $n = 1, 2, 3, \ldots$, and $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(2) If $0 \leq a_n \leq b_n$ for $n = 1, 2, 3, \ldots$, and $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty} b_n$ is divergent.

It is essential to get the inequalities in the right direction. For example, if $0 \leq a_n \leq b_n$ for $n = 1, 2, 3, \ldots$, and $\sum_{n=1}^{\infty} a_n$ is convergent, this tells you nothing about $\sum_{n=1}^{\infty} b_n$.

1. Determine whether or not the series $\sum_{n=0}^{\infty} \frac{1}{3^n + 2n}$ is convergent. Be sure to show your reasoning.

Hint: The series $\sum_{n=0}^{\infty} \frac{1}{3^n}$ is convergent.

Solution: For $n = 0, 1, 2, \ldots$, we have

$0 < 3^n \leq 3^n + 2n,$

so

$0 < \frac{1}{3^n + 2n} \leq \frac{1}{3^n}.$

Since $\sum_{n=0}^{\infty} \frac{1}{3^n}$ is convergent, the Comparison Test implies that $\sum_{n=0}^{\infty} \frac{1}{3^n + 2n}$ is convergent.

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Caution: it is also true that for \( n = 0, 1, 2, \ldots \), we have 
\[
0 < \frac{1}{2 \cdot 3^n} \leq \frac{1}{3^n + 2^n},
\]
and that \( \sum_{n=0}^{\infty} \frac{1}{2 \cdot 3^n} \) is convergent. However, this tells you nothing about convergence of \( \sum_{n=0}^{\infty} \frac{1}{3^n + 2^n} \). Be sure to get the inequalities right!

2. Determine whether or not the series \( \sum_{n=1}^{\infty} \frac{5 + 3 \sin(n)}{\sqrt{n}} \) is convergent. Be sure to show your reasoning.

\[\text{Hint: The series } \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ is divergent.}\]

\[\text{Solution: For } n = 1, 2, 3, \ldots, \text{ we have } 3 \sin(n) \geq -3, \text{ so } 0 < 2 \leq 5 + 3 \sin(n), \]
whence 
\[
0 < \frac{2}{\sqrt{n}} \leq \frac{5 + 3 \sin(n)}{\sqrt{n}}.
\]
Since \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \) is divergent, so is \( \sum_{n=1}^{\infty} \frac{2}{\sqrt{n}} \). The Comparison Test now implies that \( \sum_{n=1}^{\infty} \frac{5 + 3 \sin(n)}{\sqrt{n}} \) is divergent.

Caution: it is also true that for \( n = 1, 2, 3, \ldots \), we have 
\[
0 < \frac{5 + 3 \sin(n)}{\sqrt{n}} \leq \frac{8}{\sqrt{n}},
\]
and that \( \sum_{n=1}^{\infty} \frac{8}{\sqrt{n}} \) is divergent. However, this tells you nothing about divergence of \( \sum_{n=1}^{\infty} \frac{5 + 3 \sin(n)}{\sqrt{n}} \). Be sure to get the inequalities right!

3. Determine whether or not the series \( \sum_{n=1}^{\infty} \frac{\arctan(n^3)}{n^3} \) is convergent. Be sure to show your reasoning.
Hint: The series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent.

Solution: For $n = 1, 2, 3, \ldots$, we have

$$0 \leq \arctan\left(\frac{n^3}{3}\right) \leq \frac{\pi}{2},$$

whence

$$0 < \frac{\arctan\left(\frac{n^3}{3}\right)}{n^3} \leq \frac{\pi}{2n^3}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent, so is $\sum_{n=1}^{\infty} \frac{\pi}{2n^3}$. The Comparison Test now implies that $\sum_{n=1}^{\infty} \frac{\arctan\left(\frac{n^3}{3}\right)}{n^3}$ is convergent.

Caution: it is also true that for $n = 1, 2, 3, \ldots$, we have

$$0 < \frac{\pi}{4n^3} \leq \frac{\arctan\left(\frac{n^3}{3}\right)}{n^3},$$

and that $\sum_{n=1}^{\infty} \frac{\pi}{4n^3}$ is convergent. However, this tells you nothing about convergence of $\sum_{n=1}^{\infty} \frac{\arctan\left(\frac{n^3}{3}\right)}{n^3}$. Be sure to get the inequalities right!