WORKSHEET SOLUTIONS: ROOT TEST

Recall the Root Test for convergence of the series $\sum_{n=1}^{\infty} a_n$:

1. If $\lim_{n \to \infty} \sqrt[n]{|a_n|}$ exists and is less than 1, then the series is absolutely convergent (and therefore also convergent).
2. If $\lim_{n \to \infty} \sqrt[n]{|a_n|}$ exists and is greater than 1, or if $\lim_{n \to \infty} \sqrt[n]{|a_n|} = \infty$, then the series is divergent.
3. In all other cases, that is, if $\lim_{n \to \infty} \sqrt[n]{|a_n|} = 1$ or if this limit fails to exist in any other way than diverging to $\infty$, the Root Test says nothing. Try some other test.

1. Consider the series $\sum_{n=1}^{\infty} \frac{(3n^2 + 7)^n}{(2n^2 + 57n + 29)^n}$. Can you use the Root Test to decide convergence? Why or why not? If not, can you use some other test?

Solution: The terms are all nonnegative. Trying to apply the Root Test, we calculate:

$$\lim_{n \to \infty} \sqrt[n]{\frac{(3n^2 + 7)^n}{(2n^2 + 57n + 29)^n}} = \lim_{n \to \infty} \frac{3n^2 + 7}{2n^2 + 57n + 29} = \frac{3}{2} > 1.$$

Since $\frac{3}{2} > 1$, the Root Test applies, and says that the series $\sum_{n=1}^{\infty} \frac{(3n^2 + 7)^n}{(2n^2 + 57n + 29)^n}$ diverges.

2. Consider the series $\sum_{n=1}^{\infty} \frac{(n^2 + 57n + 29)^n}{(n^2 + 7)^n}$. Can you use the Root Test to decide convergence? Why or why not? If not, can you use some other test?

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Solution: The terms are all nonnegative. Trying to apply the Root Test, we calculate:

\[
\lim_{n \to \infty} \sqrt[n]{\frac{(n^2 + 57n + 29)^n}{(n^2 + 7)^n}} = \lim_{n \to \infty} \frac{n^2 + 57n + 29}{n^2 + 7} = 1.
\]

So the Root Test tells us nothing.

In this case, for \(n = 1, 2, 3, \ldots\), we have \(n^2 + 57n + 29 \geq n^2 + 7\), so

\[
\frac{n^2 + 57n + 29}{n^2 + 7} \geq 1,
\]

so

\[
\frac{(n^2 + 57n + 29)^n}{(n^2 + 7)^n} = \left(\frac{n^2 + 57n + 29}{n^2 + 7}\right)^n \geq 1.
\]

Therefore the terms of the series don’t converge to 0, and the series \(\sum_{n=1}^{\infty} \frac{(n^2 + 57n + 29)^n}{(n^2 + 7)^n}\) diverges.

It is slightly tricky to actually calculate

\[
\lim_{n \to \infty} \frac{(n^2 + 57n + 29)^n}{(n^2 + 7)^n}.
\]

3. Consider the series \(\sum_{n=1}^{\infty} \frac{[\ln(n)]^n}{n^{n/2}}\). Can you use the Root Test to decide convergence? Why or why not? If not, can you use some other test?

Solution: The terms are all nonnegative. Trying to apply the Root Test, we calculate:

\[
\lim_{n \to \infty} \sqrt[n]{\frac{[\ln(n)]^n}{n^{n/2}}} = \lim_{n \to \infty} \frac{\ln(n)}{\left(n^{n/2}\right)^{1/n}} = \lim_{n \to \infty} \frac{\ln(n)}{n^{1/2}}.
\]

You should have enough familiarity with rates of growth of various functions by now to expect that this limit should be zero. This can be proved by using L’Hospital’s Rule on the function \(f(x) = \ln(x)/\sqrt{x}\). Since \(0 < 1\), the Root Test tells us that the series \(\sum_{n=1}^{\infty} \frac{[\ln(n)]^n}{n^{n/2}}\) converges absolutely, and therefore converges.

4. Determine all values of \(x\) for which the power series \(\sum_{n=1}^{\infty} \frac{x^n}{2^n n^2}\) is convergent. (The coefficient of \(x^0\) is 0.)
Solution: The series certainly converges when \( x = 0 \).

For \( x \neq 0 \), we begin with the Ratio Test. (The Root Test can also be used, but is slightly trickier. You need to use the fact that \( \lim_{n \to \infty} n^{1/n} = 1 \).) We have

\[
\lim_{n \to \infty} \frac{|x^{n+1}|}{|x^n|} = \lim_{n \to \infty} \left( \frac{2^n n^2 |x|^{n+1}}{2^{n+1} (n+1)^2 |x|^n} \right)
\]

\[
= \lim_{n \to \infty} \left( \frac{|x|}{2} \right) \left( \frac{n+1}{n} \right)^2 = \left( \frac{|x|}{2} \right) \cdot 1 = \frac{|x|}{2}.
\]

So the Ratio Test implies that \( \sum_{n=1}^{\infty} \frac{x^n}{2^n n^2} \) is absolutely convergent, and hence convergent, if \( |x| < 2 \), and is divergent if \( |x| > 2 \).

It remains to test what happens when \( x = 2 \) and \( x = -2 \). For \( x = 2 \), we get

\[
\sum_{n=1}^{\infty} \frac{2^n}{2^n n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}.
\]

We know this series is convergent, since it has the form \( \sum_{n=1}^{\infty} \frac{1}{n^p} \), with exponent \( 2 > 1 \).

For \( x = -2 \), we get

\[
\sum_{n=1}^{\infty} \left| \frac{(-2)^n}{2^n n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}.
\]

As we just saw, the series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \), is convergent. So the series \( \sum_{n=1}^{\infty} \frac{(-2)^n}{2^n n^2} \) is absolutely convergent, and hence convergent.

Conclusion: the power series \( \sum_{n=1}^{\infty} \frac{x^n}{2^n n^2} \) is convergent exactly when \( x \) is in \([-2, 2]\).