MATH 253 (PHILLIPS): SOLUTIONS TO HOMEWORK 6.

This homework sheet is due in class on Wednesday 17 May 2023 (week 6).

All the requirements in the sheet on general instructions for homework apply. In particular, show your work (unlike WeBWorK as done in previous courses), give exact answers (not decimal approximations), and use correct notation. (See the web page on notation.)

1. (4 points.) Determine whether the series \( \sum_{n=1}^{\infty} \frac{n^2}{2^n} \) is absolutely convergent. Be sure to show your reasoning in mathematically and notationally correct steps.

Solution: We can use the Ratio Test. We have

\[
\lim_{n \to \infty} \left| \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} \right| = \lim_{n \to \infty} \frac{(n+1)^2}{2n^2} \cdot \frac{1}{1} = \lim_{n \to \infty} \frac{1}{2} \left( 1 + \frac{1}{n} \right)^2 = \frac{1}{2}.
\]

Since \( \frac{1}{2} < 1 \), the series \( \sum_{n=1}^{\infty} \frac{n^2}{2^n} \) is absolutely convergent.

2. (5 points.) Determine whether the series \( \sum_{n=1}^{\infty} \frac{12^n}{(2n+1)^n} \) is convergent. Be sure to show your reasoning in mathematically and notationally correct steps.

Solution: We use the Root Test. We have

\[
\lim_{n \to \infty} \sqrt[n]{\frac{12^n}{(2n+1)^n}} = \lim_{n \to \infty} \frac{12}{2n+1} = 0.
\]

Since 0 < 1, the series \( \sum_{n=1}^{\infty} \frac{12^n}{(2n+1)^n} \) is absolutely convergent. Therefore this series converges.

3. (5 points.) Determine whether the series \( \sum_{n=1}^{\infty} \frac{(-2)^n n!}{(2n)!} \) is absolutely convergent. Be sure to show your reasoning in mathematically and notationally correct steps.

Solution: We can use the Ratio Test. We have

\[
\lim_{n \to \infty} \left| \frac{(-2)^{n+1}(n+1)!}{(2n+1)!} \cdot \frac{(2n)!}{(-2)^n n!} \right| = \lim_{n \to \infty} \frac{(-2)^{n+1}(n+1)!}{(2n+1)!} \cdot \frac{(2n)!}{(-2)^n n!} \cdot \frac{(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{(2n+1)!} = \lim_{n \to \infty} \frac{1}{2} \left( 1 + \frac{1}{n} \right) = \frac{1}{2}.
\]

Since \( \frac{1}{2} < 1 \), the series \( \sum_{n=1}^{\infty} \frac{(-2)^n n!}{(2n)!} \) is absolutely convergent.

Date: 17 May 2023.
4. (5 points.) Determine whether the series \( \sum_{n=1}^{\infty} \frac{(-19)^n}{3n^2} \) is absolutely convergent. Be sure to show your reasoning in mathematically and notationally correct steps.

**Solution:** We use the Root Test. We have
\[
\lim_{n \to \infty} \sqrt[n]{ \left| \frac{(-19)^n}{3n^2} \right| } = \lim_{n \to \infty} \sqrt[n]{ \frac{19^n}{3n^2} } = \lim_{n \to \infty} \frac{19}{3n} = 0.
\]
Since \( 0 < 1 \), the series \( \sum_{n=1}^{\infty} \frac{(-19)^n}{3n^2} \) is absolutely convergent.

5. (5 points.) Determine whether the following series is absolutely convergent:
\[
1 - \frac{1 \cdot 3}{3!} + \frac{1 \cdot 3 \cdot 5}{5!} - \frac{1 \cdot 3 \cdot 5 \cdot 7}{7!} + \cdots + \frac{(-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2n-1)!} + \cdots.
\]
Be sure to show your reasoning in mathematically and notationally correct steps.

**Solution:** Write the series as
\[
\sum_{n=0}^{\infty} \frac{(-1)^{n+1} \cdot 1 \cdot 3 \cdots (2n-1) \cdot (2n+1)}{(2n+1)!}.
\]
We can use the Ratio Test. We have
\[
\lim_{n \to \infty} \left| \frac{(-1)^{(n+1)+1} \cdot 1 \cdot 3 \cdots (2n-1) \cdot (2n+1) \cdot (2(n+1)+1)!}{(2n+1)!} \right|
\]
\[
= \lim_{n \to \infty} \left( \frac{2(n+1)!}{1 \cdot 3 \cdots (2n-1) \cdot (2n+1)} \right) \left( \frac{1 \cdot 3 \cdots (2n-1) \cdot (2n+1) \cdot (2n+3)}{(2n+3)!} \right)
\]
\[
= \lim_{n \to \infty} \frac{2(n+3)}{(2n+3)} \frac{(2n+1)!}{(2n+3)!} = \lim_{n \to \infty} \frac{2(n+3)}{(2n+3) \cdot (2n+1)} = \lim_{n \to \infty} \frac{2}{2n+3} = 0.
\]
Since \( 0 < 1 \), the series \( \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \cdot 1 \cdot 3 \cdots (2n-1) \cdot (2n+1)}{(2n+1)!} \) is absolutely convergent.

6. (8 points.) Find the radius of convergence and the interval of convergence of the power series \( \sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2 + 1} \).
Be sure to show your reasoning in mathematically and notationally correct steps.

**Solution:** For \( x \neq 2 \), we test convergence using the Ratio Test. We have
\[
\lim_{n \to \infty} \left| \frac{(x-2)^{n+1}}{(n+1)^2 + 1} \right| = \lim_{n \to \infty} \left| \frac{(x-2)^n}{n^2 + 1} \right| \left( \frac{n^2 + 1}{(n+1)^2 + 1} \right) = \lim_{n \to \infty} |x-2| \left( \frac{n^2 + 1}{(n+1)^2 + 1} \right)
\]
\[
= \lim_{n \to \infty} |x-2| \left( \frac{1 + \frac{1}{n^2}}{1 + \frac{2}{n} + \frac{2}{n^2}} \right) = |x-2|.
\]
Thus, if \( |x-2| < 1 \), the series \( \sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2 + 1} \) is absolutely convergent, and hence convergent. If \( |x-2| < 1 \), the series \( \sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2 + 1} \) diverges. It follows that the radius of convergence is 1.

It remains to test for convergence at \( x = 1 \) and \( x = 3 \). For both of these, we prove absolute convergence using the Comparison Test, so that we get convergence at both endpoints.
For $x = 1$, we get
\[ \sum_{n=0}^{\infty} \left| \frac{(1-2)^n}{n^2+1} \right| = \sum_{n=0}^{\infty} \left| \frac{(-1)^n}{n^2+1} \right| = \sum_{n=0}^{\infty} \frac{1}{n^2+1}. \]

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges because the exponent 2 is greater than 1. Since $0 \leq \frac{1}{n^2+1} \leq \frac{1}{n^2}$, the Comparison Test implies that the series $\sum_{n=0}^{\infty} \left| \frac{(x-2)^n}{n^2+1} \right|$ converges absolutely when $x = 1$.

For $x = 3$, we get
\[ \sum_{n=0}^{\infty} \left| \frac{(3-2)^n}{n^2+1} \right| = \sum_{n=0}^{\infty} \frac{1}{n^2+1}. \]

We just saw that this series converges. So $\sum_{n=0}^{\infty} \left| \frac{(x-2)^n}{n^2+1} \right|$ converges absolutely when $x = 3$.

The interval of convergence is therefore $[1, 3]$.

Convergence of $\sum_{n=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{n^2+1}$ can also be proved using the Limit Comparison Test or the Integral Test.

7. (8 points.) Find the radius of convergence and the interval of convergence of the power series $\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1}$.

Be sure to show your reasoning in mathematically and notationally correct steps.

**Solution:** For $x \neq 3$, we test convergence using the Ratio Test. We have
\[
\lim_{n \to \infty} \frac{\left| \frac{(x-3)^{n+1}}{2(n+1)+1} \right|}{\left| \frac{(x-3)^n}{2n+1} \right|} = \lim_{n \to \infty} \left| \frac{x-3}{2n+1} \right| \left( \frac{2n+1}{2(n+1)+1} \right) = \lim_{n \to \infty} \left| x-3 \right| \left( \frac{2n+1}{2n+3} \right)
\]
\[= \left| x-3 \right| \lim_{n \to \infty} \frac{2n+1}{2n+3} = \left| x-3 \right|. \]

Thus, if $|x-3| < 1$, the series $\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1}$ is absolutely convergent, and hence convergent. If $|x-3| < 1$, the series $\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1}$ diverges. It follows that the radius of convergence is 1.

It remains to test for convergence at $x = 2$ and $x = 4$. For $x = 2$, we get
\[ \sum_{n=0}^{\infty} (-1)^n \frac{(2-3)^n}{2n+1} = \sum_{n=0}^{\infty} \frac{1}{2n+1} \]

We use the Limit Comparison Test to show that this series diverges. The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges because the exponent involved is 1, and
\[
\lim_{n \to \infty} \frac{\left( \frac{1}{n+1} \right)}{\left( \frac{1}{2n+1} \right)} = \lim_{n \to \infty} \frac{2n+1}{n} = \lim_{n \to \infty} \left( 2 + \frac{1}{n} \right) = 2,
\]
which is in $(0, \infty)$. Therefore $\sum_{n=0}^{\infty} \frac{1}{2n+1}$ diverges.

For $x = 4$, we get
\[ \sum_{n=0}^{\infty} (-1)^n \frac{(4-3)^n}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}. \]
The sequence of terms satisfies \( \lim_{n \to \infty} \frac{(-1)^n}{2n+1} = 0 \), the signs alternate, and the absolute values of the terms are nonincreasing: if \( a_n = \frac{(-1)^n}{2n+1} \), then \( a_n \geq a_{n+1} \) for every positive integer \( n \). Therefore the series \( \sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1} \) converges by the Alternating Series Test.

The interval of convergence is therefore \((2, 4]\).

Note: Parentheses are required in the expression \( \left(\frac{1}{2}\right) \), to make the intended order of the divisions unambiguous. Leaving them out is a notation mistake.

8. (8 points.) Find the radius of convergence and the interval of convergence of the power series \( \sum_{n=1}^{\infty} \frac{(-2)^nx^n}{\sqrt{n}} \).

Be sure to show your reasoning in mathematically and notationally correct steps.

Solution: For \( x \neq 0 \), we test convergence using the Ratio Test. We have

\[
\lim_{n \to \infty} \left| \frac{(-2)^{n+1}x^{n+1}}{\sqrt[n+1]{n+1}} \cdot \frac{\sqrt[n+1]{n+1}}{(-2)^nx^n} \right| = \lim_{n \to \infty} \left| \frac{(-2)\sqrt[n]{n+1}|x|}{\sqrt[n]{n}} \right| = \lim_{n \to \infty} 2|x| \left( \frac{\sqrt[n]{n+1}}{\sqrt[n]{n}} \right) = 2|x| \lim_{n \to \infty} \sqrt[n]{\frac{n+1}{n+1}} = 2|x| \lim_{n \to \infty} \sqrt[4]{1 + \frac{1}{n}} = 2|x| \sqrt[4]{1} = 2|x|.
\]

Thus, if \( |x| < \frac{1}{2} \), then \( 2|x| < 1 \), so the series \( \sum_{n=1}^{\infty} \frac{(-2)^nx^n}{\sqrt{n}} \) is absolutely convergent, and hence convergent. If \( |x| > \frac{1}{2} \), then \( 2|x| > 1 \), so the series \( \sum_{n=1}^{\infty} \frac{(-2)^nx^n}{\sqrt{n}} \) diverges. It follows that the radius of convergence is \( \frac{1}{2} \).

It remains to test for convergence at \( x = -\frac{1}{2} \) and \( x = \frac{1}{2} \). For \( x = -\frac{1}{2} \), we get

\[
\sum_{n=1}^{\infty} \frac{(-2)^n \left(\frac{-1}{2}\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/4}}.
\]

This series diverges because the exponent \( \frac{1}{4} \) is less than 1.

For \( x = \frac{1}{2} \), we get

\[
\sum_{n=1}^{\infty} \frac{(-2)^n \left(\frac{1}{2}\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/4}}.
\]

The sequence of terms satisfies \( \lim_{n \to \infty} \frac{(-1)^n}{n^{1/4}} = 0 \), the signs alternate, and the absolute values of the terms are nonincreasing: if \( a_n = \frac{(-1)^n}{n^{1/4}} \), then \( a_n \geq a_{n+1} \) for every positive integer \( n \). Therefore the series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/4}} \) converges by the Alternating Series Test.

The interval of convergence is therefore \((-\frac{1}{2}, \frac{1}{2}]\).

9. (8 points.) Find the radius of convergence and the interval of convergence of the power series \( \sum_{n=1}^{\infty} \frac{(4x+1)^n}{n^2} \).

(Why is this series a power series?) Be sure to show your reasoning in mathematically and notationally correct steps.

Solution: To see that this expression is a power series, rewrite it as

\[
\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(4 \left(x + \frac{1}{4}\right))^n}{n^2}.
\]
Thus, this is a power series centered at $-\frac{1}{4}$.

For $x \neq -\frac{1}{4}$, we test convergence using the Ratio Test. We have

$$\lim_{n \to \infty} \left| \frac{(4x+1)^{n+1}}{(n+1)^n} \right| = \lim_{n \to \infty} \frac{|4x+1|^{n+1}}{|n|^{n+1}} = \lim_{n \to \infty} |4x+1| \left( \frac{n^2}{(n+1)^2} \right)$$

$$= |4x+1| \lim_{n \to \infty} \left( \frac{n}{n+1} \right)^2 = |4x+1| \left( \lim_{n \to \infty} \frac{1}{1 + \frac{1}{n}} \right)^2 = |4x+1|.$$

Thus, if $|4x+1| < 1$, which is the same as $|x + \frac{1}{4}| < \frac{1}{4}$, the series $\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n^2}$ is absolutely convergent, and hence convergent. If $|4x+1| > 1$, which is the same as $|x + \frac{1}{4}| > \frac{1}{4}$, the series $\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n^2}$ diverges. It follows that the radius of convergence is $\frac{1}{4}$.

It remains to test for convergence at $x = -\frac{1}{2}$ and $x = 0$. For $x = -\frac{1}{2}$, we get

$$\sum_{n=1}^{\infty} \frac{(4 \cdot \frac{1}{2} + 1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges because the exponent 2 is greater than 1. Therefore the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolutely convergent, and hence convergent.

For $x = 0$, we get

$$\sum_{n=1}^{\infty} \frac{(4 \cdot 0 + 1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

This series converges because the exponent 2 is greater than 1.

The interval of convergence is therefore $[-\frac{1}{4}, 0]$.

10. (5 points.) Suppose the power series $\sum_{n=0}^{\infty} c_n x^n$ has radius of convergence equal to 2, and the power series $\sum_{n=0}^{\infty} d_n x^n$ has radius of convergence equal to 3. Determine the radius of convergence of the series $\sum_{n=0}^{\infty} (c_n + d_n) x^n$. Be sure to show your reasoning in mathematically and notationally correct steps.

**Solution:** Let $R$ be the radius of convergence of the series

$$(1) \quad \sum_{n=0}^{\infty} (c_n + d_n) x^n.$$

The series $\sum_{n=0}^{\infty} c_n x^n$ converges when $|x| < 2$. The series $\sum_{n=0}^{\infty} d_n x^n$ converges when $|x| < 3$; in particular, this series converges when $|x| < 2$. Therefore the series

$$\sum_{n=0}^{\infty} (c_n + d_n) x^n = \sum_{n=0}^{\infty} (c_n x^n + d_n x^n)$$

converges when $|x| < 2$. (The sum is

$$\sum_{n=0}^{\infty} (c_n + d_n) x^n = \sum_{n=0}^{\infty} c_n x^n + \sum_{n=0}^{\infty} d_n x^n,$$

but we don’t need this.) Since the series (1) converges when $|x| < 2$, its radius of convergence $R$ satisfies $R \geq 2$. 
Suppose \( 2 < |x| < 3 \). Then the series \( \sum_{n=0}^{\infty} c_n x^n \) diverges and the series \( \sum_{n=0}^{\infty} d_n x^n \) converges. We claim that the series \( (1) \) diverges. If this series converged, then, since \( \sum_{n=0}^{\infty} d_n x^n \) also converges, the series

\[
\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} [(c_n - d_n)x^n - d_n x^n]
\]

would also converge. (The sum would be

\[
\sum_{n=0}^{\infty} [(c_n - d_n)x^n - d_n x^n] = \sum_{n=0}^{\infty} (c_n + d_n)x^n - \sum_{n=0}^{\infty} d_n x^n,
\]

but we don’t need this.) This is a contradiction. So the series \( (1) \) diverges.

In particular, whenever \( 2 < |x| < 3 \), then the series \( (1) \) diverges, so \( R \leq x \). It follows that \( R \leq 2 \). Putting this together with the previous bound, we get \( R = 2 \).

A complete solution must show both \( R \geq 2 \) and \( R \leq 2 \).

The argument that the series \( (1) \) diverges when \( 2 < |x| < 3 \) is a special case of the general fact: if \( \sum_{n=0}^{\infty} a_n \) converges, and \( \sum_{n=0}^{\infty} b_n \) diverges, then \( \sum_{n=0}^{\infty} (a_n + b_n) \) diverges. The proof of this fact is the same as above.

However, if both \( \sum_{n=0}^{\infty} a_n \) and \( \sum_{n=0}^{\infty} b_n \) diverge, then no conclusion about \( \sum_{n=0}^{\infty} (a_n + b_n) \) is possible: this series could converge and also could diverge. In particular, argument that the series \( (1) \) diverges when \( 2 < |x| < 3 \) is not valid then \( |x| \geq 3 \).

11. (5 points.) Suppose the power series \( \sum_{n=0}^{\infty} c_n x^n \) has radius of convergence equal to \( R \). Determine the radius of convergence of the series \( \sum_{n=0}^{\infty} c_n x^{2n} \). Be sure to show your reasoning in mathematically and notationally correct steps.

**Solution:** Let \( S \) be the radius of convergence of the series

\[
\sum_{n=0}^{\infty} c_n x^{2n}.
\]

Then \( S \) is completely determined by the statement that the series \( (2) \) converges when \( |x| < S \) and diverges when \( |x| > S \). We claim that \( S = \sqrt{R} \) works.

Suppose first that \( |x| < \sqrt{R} \). Then \( |x^2| < R \). Therefore the series \( \sum_{n=0}^{\infty} c_n (x^2)^n \) converges. This just says that the series \( (2) \) converges. Now suppose that \( |x| > \sqrt{R} \). Then \( |x^2| > R \). Therefore the series \( \sum_{n=0}^{\infty} c_n (x^2)^n \) diverges. This just says that the series \( (2) \) diverges.

We have shown that \( S = \sqrt{R} \).

A complete solution must show both that the series \( (2) \) converges when \( |x| < \sqrt{R} \) and that this series diverges when \( |x| > \sqrt{R} \).

12. (8 points.) Find a power series representation centered at 0 for the function \( f(x) = \frac{2}{3-x} \), and determine its interval of convergence. Be sure to show your reasoning in mathematically and notationally correct steps.
Solution: We start with the known series expansion (the geometric series)

\[
\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n
\]

when \(|r| < 1\), and the fact that the series (3) diverges when \(|r| \geq 1\).

Now write

\[
\frac{2}{3-x} = \frac{\left(\frac{2}{3}\right)}{1 - \frac{x}{3}}
\]

When \(\left|\frac{x}{3}\right| < 1\), which is the same thing as \(|x| < 3\), by putting \(r = \frac{x}{3}\) in (3), we get

\[
\frac{1}{1-\frac{x}{3}} = \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{3^n}\right) x_n;
\]

moreover, this series diverges when \(|x| \geq 3\). Therefore

\[
\frac{2}{3-x} = \sum_{n=0}^{\infty} \frac{2}{3} \left(\frac{1}{3^n}\right) x^n = \sum_{n=0}^{\infty} \left(\frac{2}{3^{n+1}}\right) x^n,
\]

with interval of convergence \((-3,3)\).

One can also compute the interval of convergence of the series (6) after getting it, using the ratio test when \(0 < |x| < 3\) and \(|x| > 3\), and checking that the terms don’t converge to 0 when \(x = \pm 3\). The method used in the solution is faster.

13. (8 points.) Find a power series representation centered at 0 for the function \(f(x) = \frac{x}{2x^2+1}\), and determine its interval of convergence. Be sure to show your reasoning in mathematically and notationally correct steps.

Solution: We start with the known series expansion (the geometric series)

\[
\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n
\]

when \(|r| < 1\), and the fact that the series (5) diverges when \(|r| \geq 1\).

Now write

\[
\frac{x}{2x^2 + 1} = \frac{x}{2} \left(\frac{1}{1 - \left(-\frac{x^2}{2}\right)}\right).
\]

When \(\left|\frac{x^2}{2}\right| < 1\), which is the same thing as \(|x| < \sqrt{2}\), by putting \(r = -\frac{x^2}{2}\) in (5), we get

\[
\frac{1}{1-\left(-\frac{x^2}{2}\right)} = \sum_{n=0}^{\infty} \left(-\frac{x^2}{2}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{2^n}\right)^n x^{2n};
\]

moreover, this series diverges when \(|x| \geq \sqrt{2}\). Therefore

\[
\frac{x}{2x^2 + 1} = \sum_{n=0}^{\infty} \frac{x}{2} \left(\frac{1}{2^n}\right)^n x^{2n} = \sum_{n=0}^{\infty} \left(\frac{1}{2^{n+1}}\right)^n x^{2n+1},
\]

with interval of convergence \((-\sqrt{2}, \sqrt{2})\). Note that the right hand side is in fact a power series.

One can also compute the interval of convergence of the series (6) after getting it, using the ratio test when \(0 < |x| < \sqrt{2}\) and \(|x| > \sqrt{2}\), and checking that the terms don’t converge to 0 when \(x = \pm \sqrt{2}\). The method used in the solution is faster.

14. (8 points.) Find a power series representation for the function \(f(x) = \frac{1+x}{1-x^2}\) and determine its interval of convergence. Be sure to show your reasoning in mathematically and notationally correct steps.
Solution: We start with the known series expansion (the geometric series)

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n
\]

when \(|x| < 1\). Therefore, when \(|x| < 1\), we have

\[
\frac{1+x}{1-x} = (1+x) \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^n + x \sum_{n=0}^{\infty} x^n = (1 + x + x^2 + x^3 + \cdots) + (x + x^2 + x^3 + x^4 + \cdots)
\]

\[
= 1 + 2x + 2x^2 + 2x^3 + 2x^4 + \cdots = 1 + 2 \sum_{n=1}^{\infty} x^n.
\]

We know that the series \(\sum_{n=0}^{\infty} x^n\) converges if and only if \(|x| < 1\). Therefore the same is true of \(\sum_{n=1}^{\infty} x^n\), and so also of \(1 + 2 \sum_{n=1}^{\infty} x^n\). So the interval of convergence is \((-1, 1)\).

Alternate solution: We have

\[
\frac{1+x}{1-x} = \frac{2}{1-x} - 1 = 2 \sum_{n=0}^{\infty} x^n - 1 = 1 + 2 \sum_{n=1}^{\infty} x^n.
\]

The argument for the interval of convergence is the same as in the first solution.

One of the expressions

\[
1 + 2x + 2x^2 + 2x^3 + 2x^4 + \cdots \quad \text{or} \quad 1 + 2 \sum_{n=1}^{\infty} x^n
\]

must appear in any solution; otherwise, the supposed solution doesn’t exhibit a power series.

15. (5 points.) Use the definition of the Taylor series to find the Taylor series centered at 0 for the function \(f(x) = \ln(1+x)\), and determine the radius of convergence of the series. (Don’t try to show that the series converges to \(f(x)\). That is, don’t attempt to estimate the remainder \(R_n(x)\).) Be sure to show your reasoning in mathematically and notationally correct steps.

Solution: We have \(f(0) = 0\). We calculate derivatives:

\[
f'(x) = (1+x)^{-1} \quad \text{and} \quad f'(0) = 1,
\]

\[
f''(x) = -(1+x)^{-2} \quad \text{and} \quad f''(0) = -1,
\]

\[
f'''(x) = 2(1+x)^{-3} \quad \text{and} \quad f'''(0) = -2,
\]

\[
f''''(x) = -3 \cdot 2 \cdot (1+x)^{-4} \quad \text{and} \quad f''''(0) = -3 \cdot 2,
\]

and, in general,

\[
f^{(n)}(x) = (-1)^n (n-1)! (1+x)^{-n} \quad \text{and} \quad f^{(n)}(0) = (-1)^n (n-1) !.
\]

Therefore the series is

\[
\sum_{n=1}^{\infty} \left( \frac{f^{(n)}(0)}{n!} \right) x^n = \sum_{n=1}^{\infty} (-1)^n \frac{(n-1)!}{n!} x^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n.
\]

16. (5 points.) Use the definition of the Taylor series to find the Taylor series centered at 0 for the function \(f(x) = \cos(3x)\), and determine the radius of convergence of the series. (Don’t try to show that the series converges to \(f(x)\). That is, don’t attempt to estimate the remainder \(R_n(x)\).) Be sure to show your reasoning in mathematically and notationally correct steps.
Solution: We calculate derivatives:

\[
\begin{align*}
  f(x) &= \cos(3x) \quad \text{and} \quad f(0) = 1, \\
  f'(x) &= -3 \sin(3x) \quad \text{and} \quad f'(0) = 0, \\
  f''(x) &= -9 \cos(3x) \quad \text{and} \quad f''(0) = -3^2, \\
  f'''(x) &= 27 \sin(3x) \quad \text{and} \quad f'''(0) = 0, \\
  f''''(x) &= 3^4 \cos(3x) \quad \text{and} \quad f''''(0) = 3^4.
\end{align*}
\]

and

\[
\begin{align*}
  f''''(x) &= 3^4 \cos(3x) = 3^4 f(x) \quad \text{and} \quad f''''(0) = 3^4.
\end{align*}
\]

Generally,

\[
\begin{align*}
  f^{(n+4)}(x) &= 3^4 f^{(n)}(x) \quad \text{and} \quad f^{(n+4)}(0) = 3^4 f^{(n)}(0).
\end{align*}
\]

We can write the values as \( f^{(n)}(0) = 0 \) when \( n \) is odd and \( f^{(2n)}(0) = (-1)^n 3^{2n} \). Therefore the series is

\[
\sum_{n=0}^{\infty} \left( \frac{f^{(n)}(0)}{n!} \right) x^n = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n}}{(2n)!} x^{2n}.
\]