WORKSHEET SOLUTIONS: CONVERGENCE OF THE TAYLOR SERIES FOR $\ln(1 + x)$ AND $\arctan(x)$

Names and student IDs: Solutions $\pi \pi \pi - \pi - \pi \pi \pi$

Recall: On its open interval of convergence (that is, excluding the endpoints), a power series can be integrated and differentiated “term by term” (or “summand by summand”). That is, if $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges to $f(x)$ for $x$ in $(a-R, a+R)$ ($R = \infty$ is allowed, which means $(-\infty, \infty)$ regardless of $a$), then $f$ is differentiable on $(a - R, a + R)$, and:

1. For every $x$ in $(a - R, a + R)$, we have
   \[
   \int_a^x f(t) \, dt = \sum_{n=0}^{\infty} \frac{c_n(x-a)^{n+1}}{n+1}.
   \]

2. For every $x$ in $(a - R, a + R)$, we have
   \[
   f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}.
   \]

Warnings: nothing is said about what happens at $a + R$ and $a - R$, and these statements are often false for other kinds of series. (See the lecture discussion for more.)

1. Write a power series for $f(x) = \frac{1}{1 + x}$. For which values of $x$ does this series converge to $\frac{1}{1 + x}$? How do you know?

   Solution: We know from our study of geometric series that if $|r| < 1$ then $\sum_{n=0}^{\infty} r^n$ converges to $\frac{1}{1 - r}$. Put $r = -x$, getting $\frac{1}{1 + x} = \sum_{n=0}^{\infty} (-1)^n x^n$ whenever $|x| < 1$ (since $|-x| = |r|$).

Date: 18 May 2023.
**Alternate solution (sketch):** Get the series \( \sum_{n=0}^{\infty} (-1)^n x^n \) from the binomial expansion of \((1 + x)^{-1}\). You still need the results on geometric series to show that this series actually converges to \( \frac{1}{1 + x} \) when \(|x| < 1\).

2. Given your solution to Problem 1, write a power series for \( F(x) = \ln(1+x) \). On which open interval does your series converge to \( \ln(1+x) \)? How do you know? (Ignore the endpoints.) (This series is in the book. We are doing it here because it is one of the basic series you need to be familiar with.)

**Solution:** Write

\[
\ln(1 + x) = \int_0^x \frac{1}{1 + t} \, dt.
\]

Then integrate the series in Problem 1 term by term, justified by (1) above. The result is that the series \( \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} \) converges to \( \ln(1+x) \) for \( x \) in \((-1, 1)\) (and maybe or maybe not also at \( x = -1 \) or \( x = 1 \)). The series can be written in nicer form as \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \).

(It turns out that the series diverges for \( x = -1 \) and converges to \( \ln(2) \) for \( x = 1 \).)

3. Write a power series for \( g(x) = \frac{1}{1 + x^2} \). For which values of \( x \) does this series converge to \( \frac{1}{1 + x^2} \)? How do you know?

**Solution:** We know from Problem 1 that \( \frac{1}{1 + z} = \sum_{n=0}^{\infty} (-1)^n z^n \) whenever \(|z| < 1\). If \(|x| < 1\) then \(|x^2| < 1\), so the series \( \sum_{n=0}^{\infty} (-1)^n x^{2n} \) converges to \( \frac{1}{1 + x^2} \) whenever \(|x| < 1\). This series diverges for all other values of \( x \), because the summands don’t approach zero.
4. Given your solution to Problem 3, write a power series for \( G(x) = \arctan(x) \). On which open interval does your series converge to \( \arctan(x) \)? How do you know? (Ignore the endpoints.)

(This series is in the book. We are doing it here because it is one of the basic series you need to be familiar with.)

\[ \text{Solution: Write} \]
\[ \arctan(x) = \int_0^x \frac{1}{1 + t^2} \, dt. \]

Then integrate the series in Problem 2 term by term, justified by (1) above. The result is that the series \( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \) converges to \( \arctan(x) \) for \( x \) in \((-1, 1)\) (and maybe or maybe not also at \( x = -1 \) or \( x = 1 \)).

(It turns out that the series converges to \( \arctan(x) \) also for \( x = -1 \) and \( x = 1 \).)

5. Find, with justification, a power series centered at 0 which converges on \((-1, 1)\) to the function

\[ H(x) = \int_0^x \frac{1}{1 + t^6} \, dt. \]

\[ \text{Solution (sketch): By the same reasoning as in Problems 3 and 4,} \]
\[ \text{the series } \sum_{n=0}^{\infty} (-1)^n x^{6n} \text{ converges to } \frac{1}{1 + x^6} \text{ for all } x \text{ in } (-1, 1), \text{ and the} \]
\[ \text{series } \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+1}}{6n + 1} \text{ converges to } \int_0^x \frac{1}{1 + t^6} \, dt \text{ for all } x \text{ in } (-1, 1). \]

6. Use your answer to Problem 5 to approximate \( \int_0^{0.1} \frac{1}{1 + t^6} \, dt \) to within \( 10^{-10} \).

\[ \text{Solution: Use the series } \sum_{n=0}^{\infty} \frac{(-1)^n (0.1)^{6n+1}}{6n + 1}. \text{ We verify the conditions for the error estimate in the Alternating Series Test. (I must see you do this!)} \]
It is clear from the definition that the signs of the summands alternate. The absolute values of the summands are the numbers \( b_n = \frac{(0.1)^{6n+1}}{6n+1} \) for \( n = 0, 1, 2, \ldots \). Since \( (0.1)^{6n+1} \) is decreasing and positive, and \( 6n+1 \) is increasing and positive, clearly \( (b_n)^\infty_{n=0} \) is decreasing. Also \( 0 \leq b_n \leq (0.1)^{6n+1} \), and \( \lim_{n \to \infty} (0.1)^{6n+1} = 0 \), so \( \lim_{n \to \infty} b_n = 0 \) by the Squeeze Theorem.

We have verified the three conditions for the error estimate in the Alternating Series Test. Therefore the error is less than the absolute value of the first term not used. Clearly

\[
b_2 = \frac{(0.1)^{6\cdot 2+1}}{6 \cdot 2 + 1} < 10^{-13} < 10^{-10},
\]

so we only need to use

\[
1 + \frac{(-1)^n (0.1)^{6\cdot 1 + 1}}{6 \cdot 1 + 1} = 1 - \frac{10^{-7}}{7}.
\]

Bonus problem. Explain why it is not possible for a power series centered at 0 to converge to \( f(x) = \frac{1}{1+x} \) on the interval \((-1,1)\) and also at \( x = -2 \).

Hint: Consider the behavior of \( f(x) \) near \( x = -1 \).

Solution: If a series \( \sum_{n=0}^{\infty} c_n x^n \) converges at \(-2\), then it must converge on \((-2,2)\) to a continuous function \( h \) defined on \((-2,2)\). Since \( \lim_{x \to -1^+} \frac{1}{1+x} = \infty \), there is no continuous function \( h \) defined on \((-2,2)\) which satisfies \( h(x) = \frac{1}{1+x} \) for all \( x \) in \((-1,1)\).