1. Define a function $f$ by

$$f(x) = \begin{cases} 
\frac{e^{2x} - 1}{x} & x \neq 0 \\
2 & x = 0.
\end{cases}$$

Find, with justification, a power series centered at 0 which converges to $f(x)$ for all real numbers $x$. Be sure to show your reasoning in mathematically and notationally correct steps.

2. Define a function $f$ by

$$f(x) = \begin{cases} 
\sin(x) & x \neq \pi \\
-1 & x = \pi.
\end{cases}$$

Find, with justification, a power series centered at $\pi$ which converges to $f(x)$ for all real numbers $x$. Be sure to show your reasoning in mathematically and notationally correct steps.

3. Define a function $c$ by

$$c(x) = \begin{cases} 
1 - \cos(x) & x \neq 0 \\
\frac{1}{2} & x = 0.
\end{cases}$$

Find $c^{(4)}(0)$. Give justification. Be sure to show your reasoning in mathematically and notationally correct steps.

4. Define a function $q$ by

$$q(x) = \begin{cases} 
\sqrt[3]{x} - 1 & x \neq 1 \\
\frac{1}{3} & x = 1.
\end{cases}$$

Find $q'(1)$. Give justification. Be sure to show your reasoning in mathematically and notationally correct steps.

5. Explain why there is a differentiable function $f$ defined for all real numbers $x$ such that $h(x) = \cos(x^{7/2})$ when $x \geq 0$. Then find $h'(0)$. Give justification. Be sure to show your reasoning in mathematically and notationally correct steps.

Date: 24 May 2023.
6. Find a number \( d \) such that the approximation
\[
\cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}
\]
gives an error of absolute value less than 0.005 on the interval \((-d, d)\). Use Taylor’s Inequality or the error estimate from the Alternating Series Test, and give the best value of \( d \) you can using this estimate. Be sure to show your reasoning in mathematically and notationally correct steps.

7. Is it possible for a power series centered at 3 to have \((-7, 7]\) as its interval of convergence? If so, give an example, and show in mathematically and notationally correct steps that your example works. If not, explain in mathematically and notationally correct steps why not.

8. Is it possible for a power series centered at \(-3\) to have \((-6, 0)\) as its interval of convergence? If so, give an example, and show in mathematically and notationally correct steps that your example works. If not, explain in mathematically and notationally correct steps why not.

9. Suppose the function \( g \) has the power series expansion 
\[
f(x) = \sum_{n=1}^{\infty} \frac{(x - 3)^n(n+1)}{n!}
\]
for all real numbers \( x \). Using mathematically and notationally correct steps, find \( f^{(12)}(3) \).

10. (8 points.) Suppose the function \( h \) has the power series expansion 
\[
h(x) = \sum_{n=1}^{\infty} \frac{(x + 5)^n}{n^3}
\]
for \( x \) in \([-6, -4]\). Using mathematically and notationally correct steps, find \( h^{(4)}(-5) \).

11. Is it possible for a power series centered at \(-7\) to have \((-7, 7]\) as its interval of convergence? If so, give an example, and show in mathematically and notationally correct steps that your example works. If not, explain in mathematically and notationally correct steps why not.