1. (3 points.) Find the sum of the series $\sum_{n=0}^{\infty} \frac{6^n}{n!}$. Be sure to show your reasoning in mathematically and notationally correct steps.

Solution: We know that for every real number $x$, the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges to $e^x$. Therefore

$$\sum_{n=0}^{\infty} \frac{6^n}{n!} = e^6.$$ 

2. (3 points.) Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n(2\pi)^{2n+1}}{(2n+1)!}$. Be sure to show your reasoning in mathematically and notationally correct steps.

Solution: We know that for every real number $x$, the series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ converges to $\sin(x)$. Therefore

$$\sum_{n=0}^{\infty} \frac{(-1)^n(2\pi)^{2n+1}}{(2n+1)!} = \sin(2\pi) = 0.$$ 

The simplification is necessary.

3. (3 points.) Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{n!}$ in terms of standard elementary functions. Be sure to show your reasoning in mathematically and notationally correct steps.

Solution: We know that for every real number $x$, the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges to $e^x$. Therefore, for every real number $x$,

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{n!} = \sum_{n=0}^{\infty} \frac{(-x^4)^n}{n!} = \exp(-x^4).$$ 

4. (3 points.) Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{6^{2n}(2n)!}$. Be sure to show your reasoning in mathematically and notationally correct steps.

Date: 9 June 2023.
Solution: We know that for every real number $x$, the series \( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \) converges to \( \cos(x) \). Therefore
\[
\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!} \cdot \left( \frac{\pi}{6} \right)^{2n} = \cos \left( \frac{\pi}{6} \right) = \frac{\sqrt{3}}{2}.
\]
The simplification is necessary.

5. (3 points.) Find the sum of the series \( \sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n}{n \cdot 5^n} \). Be sure to show your reasoning in mathematically and notationally correct steps.

Solution: We know that for every $x$ in $(-1, 1)$, the series \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \) converges to \( \ln(1+x) \). Therefore
\[
\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n}{n \cdot 5^n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n}{n} \cdot \left( \frac{3}{5} \right)^n = \ln \left( 1 + \frac{3}{5} \right) = \ln \left( \frac{8}{5} \right).
\]
The simplification is necessary.

6. (3 points.) Find the sum of the series \( \sum_{n=1}^{\infty} \frac{3^n}{5^n \cdot n!} \). Be sure to show your reasoning in mathematically and notationally correct steps.

Solution: We know that for every real number $x$, the series \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \) converges to \( e^x \). Therefore
\[
\sum_{n=1}^{\infty} \frac{3^n}{5^n \cdot n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{3}{5} \right)^n = e^{3/5}.
\]

7. (3 points.) Find the sum of the series
\[
1 - \ln(2) + \frac{[\ln(2)]^2}{2!} - \frac{[\ln(2)]^3}{3!} + \cdots.
\]
Be sure to show your reasoning in mathematically and notationally correct steps.

Solution: We know that for every real number $x$, the series \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \) converges to \( e^x \). Therefore
\[
1 - \ln(2) + \frac{[\ln(2)]^2}{2!} - \frac{[\ln(2)]^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{[- \ln(2)]^n}{n!} = e^{-\ln(2)} = \frac{1}{2}.
\]
The simplification is necessary.

8. (3 points.) Find the sum of the series
\[
\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} + \frac{1}{7 \cdot 2^7} + \cdots.
\]
Be sure to show your reasoning in mathematically and notationally correct steps. (This one is tricky. Hint: the relevant Taylor series is gotten by term by term iteration of something more recognizable.)
Solution: Consider the series $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$. Differentiating this series term by term gives $\sum_{n=0}^{\infty} x^{2n}$. We know that, for every $x$ in $(-1, 1)$, the series $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$. When $x$ is in $(-1, 1)$, then also $x^2$ is in $(-1, 1)$, so $\sum_{n=0}^{\infty} x^{2n}$ converges to $\frac{1}{1-x^2}$. Therefore, by the theorem on term by term integration of power series, for every $x$ in $(-1, 1)$, the series $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$ converges to $\int_0^x \frac{1}{1-t^2} \, dt$. Therefore

$$\frac{1}{1} + \frac{1}{3} \cdot 2 + \frac{1}{5} \cdot 2^2 + \frac{1}{7} \cdot 2^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{2n+1} \left(\frac{1}{2}\right)^{2n+1} = \int_0^{1/2} \frac{1}{1-t^2} \, dt.$$ 

Now

$$\frac{1}{1-t^2} = \frac{1}{2} \left(\frac{1}{1-t} - \frac{1}{1+t}\right)$$

so

$$\int_0^{1/2} \frac{1}{1-t^2} \, dt = \frac{1}{2} \left(\int_0^{1/2} \frac{1}{1-t} \, dt - \int_0^{1/2} \frac{1}{1+t} \, dt\right) = \frac{1}{2} \left(-\ln(1-x)\right|_0^{1/2} - \ln(1+x)\right|_0^{1/2})$$

$$= \frac{1}{2} \left(-\ln\left(1 - \frac{1}{2}\right) + \ln(1-0) + \ln\left(1 + \frac{1}{2}\right) - \ln(1-0)\right)$$

$$= -\ln\left(\frac{1}{2}\right) + \ln\left(\frac{3}{2}\right) = \frac{1}{2} \ln(2) + \ln(3) - \ln(2) = \frac{1}{2} \ln(3).$$

9. (5 points.) Find a number $d$ such that the approximation

$$\arctan(x) \approx x - \frac{x^3}{3} + \frac{x^5}{5}$$

gives an error of absolute value less than 0.005 on the interval $(-d,d)$. Use Taylor’s Inequality or the error estimate from the Alternating Series Test, and give the best value of $d$ you can using this estimate. Be sure to show your reasoning in mathematically and notationally correct steps.

Solution: This solution is based on the Alternating Series Test, for which the error is less than the absolute value of the first term not used. The terms certainly alternate in sign. If $|x| \leq 1$, then the terms and converge to zero as $n \to \infty$ and the absolute values of the terms are nonincreasing. (You must show in your solution that you checked these facts.) Therefore we will require $d \leq 1$.

Thus, if $|x| < 1$ and $x$ is in $(-d,d)$, then the error is less than $x^7/7 < d^7/7$. We thus want $d \leq 1$ and $d^7/7 \leq 0.005$. The second condition is the same as

$$d \leq \sqrt[7]{7 \cdot 0.005} = \sqrt[7]{0.035} \approx 0.61945.$$ 

Since this number is less than 1, the choice $d = 0.61944$ works.

Alternate solution (outline): The polynomial used is the 6th degree Taylor polynomial, since the coefficient of $x^6$ in the Taylor series for $\arctan(x)$ is zero. For $f(x) = \arctan(x)$, we thus need a bound on $|f^{(7)}(x)|$. The derivative is messy to calculate. A computer algebra system told me

$$f^{(7)}(x) = \frac{720(7x^6 - 35x^4 + 21x^2 - 1)}{(x^2 + 1)^7}.$$ 

We demand that $d \leq 1$, since otherwise the series does not converge. A naive estimate is that if $x$ is in $[-1,1]$ then

$$|7x^6 - 35x^4 + 21x^2 - 1| \leq 7 + 35 + 21 + 1 = 64.$$
Therefore we can take the number $M$ in Taylor’s Inequality to be

$$M = \frac{720 \cdot 64}{(d^2 + 1)^7}.$$

The rest of the solution is as usual for this sort of problem.

10. (5 points.) Find the terms through degree 4 of the general power series solution to the equation $y'(x) = xy(x) + 1$ centered at 0. Then find the terms through degree 4 of the solution which satisfies $y(0) = 1$. Be sure to show your reasoning in mathematically and notationally correct steps.

**Solution:** Assume $y(x)$ has the form $y(x) = a_0 + a_1 x + a_2 x^2 + \cdots$. Then $y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$. Substituting in $y'(x) = xy(x) + 1$ gives

$$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = 1 + a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + \cdots.$$

So we get

$$a_1 = 1, \quad a_2 = \frac{1}{2} a_0, \quad a_3 = \frac{1}{3} a_1 = \frac{1}{3}, \quad \text{and} \quad a_4 = \frac{1}{4} a_2 = \frac{1}{8} a_0.$$

Therefore

$$y(x) = a_0 + x + \frac{1}{2} a_0 x^2 + \frac{1}{3} x^3 + \frac{1}{8} a_0 x^4 + \cdots.$$

The initial value $y(0) = 1$ tells us $a_0 = 1$, and so the particular solution is

$$y(x) = 1 + x + \frac{1}{2} x^2 + \frac{1}{3} x^3 + \frac{1}{8} x^4 + \cdots.$$

11. (5 points.) Find the terms through degree 4 of the general power series solution to the equation $y'(x) = xy(x) + 1$ centered at 0. Then find the terms through degree 4 of the solution which satisfies $y(0) = 1$.

**Solution:** Assume $y(x)$ has the form $y(x) = a_0 + a_1 x + a_2 x^2 + \cdots$. Then $y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$. Substituting in $y'(x) = xy(x) + 1$ gives

$$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = 1 + a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + \cdots.$$

So we get

$$a_1 = 1, \quad a_2 = \frac{1}{2} a_0, \quad a_3 = \frac{1}{3} a_1 = \frac{1}{3}, \quad \text{and} \quad a_4 = \frac{1}{4} a_2 = \frac{1}{8} a_0.$$

Therefore

$$y(x) = a_0 + x + \frac{1}{2} a_0 x^2 + \frac{1}{3} x^3 + \frac{1}{8} a_0 x^4 + \cdots.$$

The initial value $y(0) = 1$ tells us $a_0 = 1$, and so the particular solution is

$$y(x) = 1 + x + \frac{1}{2} x^2 + \frac{1}{3} x^3 + \frac{1}{8} x^4 + \cdots.$$

12. (5 points.) Find the terms through degree 5 of the general power series solution to the equation $y''(x) + xy'(x) + y(x) = 0$ centered at 0. Then find the terms through degree 5 of the solution which satisfies $y(0) = 2$ and $y'(0) = 1$.

**Solution:** We set $y(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ and compute

$$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

and

$$y''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + \cdots.$$

Then

$$y''(x) + xy'(x) + y(x) = (2a_2 + a_0) + (6a_3 + 2a_1)x + (12a_4 + 3a_2)x^2 + (20a_5 + 4a_3)x^3 + \cdots.$$
13. (5 points.) Find the terms through degree 4 of the general power series solution to the equation \((1 - x)y''(x) + y(x) = 0\). Then find the terms through degree 4 of the particular solution which satisfies \(y(0) = 1\) and \(y'(0) = 3\).

**Solution:** If \(y(x) = \sum_{n=0}^{\infty} a_n x^n\) then \(y''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + \cdots\) and

\[
(1 - x)y''(x) + y(x) = (2a_2 + a_0) + (6a_3 - 2a_2 + a_1) x + (12a_4 - 6a_3 + a_2) x^2 + \cdots
\]

Since this must equal zero, we get the equations

\[
a_2 = -\frac{a_0}{2}, \quad a_3 = -\frac{a_1}{3}, \quad a_4 = -\frac{a_2}{4} = \frac{a_0}{8}, \quad a_5 = -\frac{a_3}{5} = \frac{a_1}{15}.
\]

The general solution is

\[
y(x) = a_0 + a_1 x - \frac{a_0}{2} x^2 - \frac{a_1}{3} x^3 - \frac{a_0}{8} x^4 + \frac{a_1}{15} x^5 + \cdots
\]

The initial values \(y(0) = 1\) and \(y'(0) = 3\) give \(a_0 = 1\) and \(a_1 = 3\), so we get

\[
y(x) = 1 + 3x - \frac{1}{2} x^2 - \frac{2}{3} x^3 - \frac{7}{24} x^4 + \cdots.
\]

14. (5 points.) Find the terms through degree 4 of the power series solution to \(x y''(x) + y'(x) + xy(x) = 0\) centered at \(x = 1\), satisfying the initial values \(y(1) = 0\) and \(y'(1) = 2\).

**Solution:** Since we are centered at 1 we write

\[
y(x) = a_0 + a_1 (x - 1) + a_2 (x - 1)^2 + a_3 (x - 1)^3 + \cdots.
\]

Then

\[
y'(x) = a_1 + 2a_2 (x - 1) + 3a_3 (x - 1)^2 + \cdots
\]

and

\[
y''(x) = 2a_2 + 6a_3 (x - 1) + 12a_4 (x - 1)^2 + 20a_5 (x - 1)^3 + \cdots
\]

In our original differential equation, we change each occurrence of \(x\) in the coefficients to \(1 + (x - 1)\). So the differential equation is

\[
[1 + (x - 1)] y''(x) + y'(x) + [1 + (x - 1)] y(x) = 0.
\]

Plugging in our formulas for \(y(x)\) and its derivatives, we get

\[
0 = (2a_2 + a_1 + a_0) + (6a_3 + 2a_2 + 2a_1 + a_0) (x - 1) + (12a_4 + 6a_3 + 3a_3 + a_2 + a_1) (x - 1)^2 + \cdots.
\]

Equating the coefficients to zero, we get the equations

\[
0 = 2a_2 + a_1 + a_0, \quad 0 = 6a_3 + 4a_2 + a_1 + a_0, \quad \text{and} \quad 0 = 12a_4 + 9a_3 + a_2 + a_1.
\]
Solving recursively, we get
\[ a_2 = -\frac{a_1 + a_0}{2}, \]
\[ a_3 = -\left(\frac{4a_2 + a_1 + a_0}{6}\right) = -\left(\frac{-2a_1 - 2a_0 + a_1 + a_0}{6}\right) = \frac{a_1 + a_0}{6}, \]
\[ a_4 = -\frac{9a_3 + a_2 + a_1}{12} = -\left(\frac{3}{2}(a_1 + a_0) - \frac{a_0 + a_1}{2} + a_1\right) = -\frac{2a_1 + a_0}{12}. \]

The general solution is therefore
\[ y(x) = a_0 + a_1(x - 1) - \left(\frac{a_0 + a_1}{2}\right)(x - 1)^2 + \left(\frac{a_0 + a_1}{6}\right)(x - 1)^3 + \left(\frac{a_0 - 2a_1}{12}\right)(x - 1)^4 + \cdots. \]

The initial values give us \( a_0 = 0 \) and \( a_1 = 2 \), so the particular solution is
\[ y(x) = 2(x - 1) - (x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{3}(x - 1)^4 + \cdots. \]

Alternate solution: Since we were not asked to find the general solution in this problem, we could save some time and reason a bit differently. We are given \( y(1) = 0 \) and \( y'(1) = 2 \). Plugging \( x = 1 \) into the differential equation gives
\[ 1 \cdot y''(1) + y'(1) + \frac{1}{2} y(1) = 0, \quad \text{or} \quad y''(1) + 2 + 0 = 0. \]
So \( y''(1) = -2 \). Taking derivatives of the differential equation gives
\[ xy''(x) + y''(x) + y''(x) + y(x) + xy'(x) = 0. \]
Plugging in \( x = 1 \) gives
\[ y''(1) + 2y''(1) + y'(1) + y(1) = 0, \quad \text{or} \quad y''(1) - 4 + 2 + 0 = 0. \]
So \( y''(1) = 2 \).

We need to go one more step to find \( y^{(4)}(1) \). Taking derivatives yet again gives
\[ xy^{(4)}(x) + y'''(x) + 2y''(x) + y'(x) + y'(x) + xy''(x) = 0. \]
Plugging in \( x = 1 \) gives
\[ y^{(4)}(1) + 3y'''(1) + 2y''(1) + y''(1) = 0, \quad \text{or} \quad y^{(4)}(1) + 6 + 4 + (-2) = 0. \]
So \( y^{(4)}(1) = -8 \).

The Taylor series for \( y(x) \) centered at \( x = 1 \) is therefore
\[ y(x) = y(1) + y'(1)(x - 1) + \frac{y''(1)}{2}(x - 1)^2 + \frac{y'''(1)}{6}(x - 1)^3 + \frac{y^{(4)}(1)}{24}(x - 1)^4 \cdots \]
\[ = 0 + 2(x - 1) - (x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{3}(x - 1)^4 + \cdots. \]

15. (4 points.) Define a function \( S \) by
\[ S(x) = \begin{cases} \sin(5x)/x & x \neq 0 \\ 0 & x = 0. \end{cases} \]
What can you say about \( S'(0) \)? Give justification.

Solution: Using either the definition of the derivative or L’Hopital’s Rule, we see that
\[ \lim_{x \to 0} \frac{\sin(5x)}{x} = 5. \]
(If you use L’Hopital’s Rule, be sure to point out that its hypotheses are satisfied, since the limit has the indeterminate form \( \frac{0}{0} \).) So \( \lim_{x \to 0} S(x) = 5 \). Since \( S(0) = 0 \), it follows that \( S \) is not continuous at 0. Therefore \( S'(0) \) does not exist.
Alternate solution: We start with the Taylor series centered at 0 for \( \sin(x) \):
\[
\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,
\]
valid for all real numbers \( x \). For all real numbers \( x \), we therefore have
\[
\sin(5x) = \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1} x^{2n+1}}{(2n+1)!} = 5x - \frac{5^3 x^3}{3!} + \frac{5^5 x^5}{5!} - \frac{5^7 x^7}{7!} + \cdots,
\]
so, if \( x \neq 0 \),
\[
S(x) = \frac{\sin(5x)}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1} x^{2n}}{(2n+1)!} = 5 - \frac{5^3 x^2}{3!} + \frac{5^5 x^4}{5!} - \frac{5^7 x^6}{7!} + \cdots.
\]
The function
\[
T(x) = 5 - \frac{5^3 x^2}{3!} + \frac{5^5 x^4}{5!} - \frac{5^7 x^6}{7!} + \cdots,
\]
being given by a power series which converges for every real number \( x \), is a continuous (in fact infinitely often differentiable) function. Since \( S(x) = T(x) \) for \( x \neq 0 \) but \( S(0) \neq T(0) \), it follows that \( S \) is not continuous at 0. Therefore \( S'(0) \) does not exist.

16. (4 points/part.) Consider the differential equation
\[
y''(t) - t^2 y(t) = 0.
\]
(a) Suppose \( y_0(t) = \sum_{n=0}^{\infty} c_n t^n \) is a solution to this equation. For \( n = 4, 5, 6, 7, \ldots \), give an equation for \( c_{n+1} \) in terms of \( c_0, c_1, \ldots, c_n \).

Solution: We have
\[
y'(t) = \sum_{n=1}^{\infty} n c_n t^{n-1} \quad \text{and} \quad y''(t) = \sum_{n=2}^{\infty} (n-1)n c_n t^{n-2},
\]
while
\[
t^2 y(t) = \sum_{n=0}^{\infty} c_n t^{n+2}.
\]
Change the indexing and apply (1) to get
\[
\sum_{n=0}^{\infty} (n+1)(n+2) c_{n+2} t^n = \sum_{n=2}^{\infty} c_{n-2} t^n.
\]
By uniqueness of the power series representation of a function, we must have
\[
(1 \cdot 2)c_2 = 0, \quad (3 \cdot 4)c_3 = 0, \quad (4 \cdot 5)c_4 = c_0,
\]
\[
(4 \cdot 5)c_5 = c_1, \quad (7 \cdot 8)c_6 = c_2, \quad (8 \cdot 9)c_7 = c_3,
\]
e etc., that is,
\[
c_{n+1} = \frac{c_{n-3}}{n(n+1)}
\]
for \( n = 4, 5, 6, 7, \ldots \).

(b) Find a power series expansion centered at 0 for the solution \( t \mapsto y_0(t) \) to (1) which satisfies \( y_0(0) = 1 \) and \( y_0'(0) = 0 \).

Solution: The initial conditions imply \( c_0 = 1 \) and \( c_1 = 0 \). From Part (a) we get \( c_2 = c_3 = 0 \). Applying the recursion relation in Part (a), we see that \( c_n = 0 \) whenever \( n \) is not a multiple of 4, and that
\[
c_0 = 1, \quad c_4 = \frac{1}{3 \cdot 4}, \quad c_6 = \frac{1}{3 \cdot 4 \cdot 7 \cdot 8}, \quad c_9 = \frac{1}{3 \cdot 4 \cdot 7 \cdot 8 \cdot 11 \cdot 12}.
\]
etc. So
\[ y_0(t) = \sum_{n=0}^{\infty} \frac{t^{4n}}{3 \cdot 4 \cdot 7 \cdot 8 \cdots (4n - 1) \cdot (4n)}. \]

(c) Find a power series expansion centered at 0 for the solution \( t \mapsto y_1(t) \) to (1) which satisfies \( y_1(0) = 0 \) and \( y'_1(0) = 1 \).

**Solution:** We assume \( y_1(t) = \sum_{n=0}^{\infty} d_n t^n \). The initial conditions tell us that \( d_0 = 0 \) and \( d_1 = 1 \). From Part (a) we get \( d_2 = d_3 = 0 \). Applying the recursion relation in Part (a), we see that \( c_n = 0 \) whenever \( n \) is not 1 plus a multiple of 4, and that
\[ d_1 = 1, \quad d_5 = \frac{1}{4 \cdot 5}, \quad d_9 = \frac{1}{4 \cdot 5 \cdot 8 \cdot 9}, \]
\[ d_{13} = \frac{1}{4 \cdot 5 \cdot 8 \cdot 9 \cdot 9 \cdot 10}, \]
etc. So the power series is
\[ \sum_{n=0}^{\infty} \frac{t^{4n+1}}{4 \cdot 5 \cdot 8 \cdot 9 \cdots (4n) \cdot (4n + 1)}. \]
(By convention, in this expression, for \( n = 0 \) there are no factors in the denominator, and it is taken to be 1. This is similar to the convention that \( 0! = 1 \).)

(d) For arbitrary constants \( a_0 \) and \( a_1 \), find a power series expansion centered at 0 for the solution \( t \mapsto y(t) \) to (1) which satisfies \( y(0) = a_0 \) and \( y'(0) = a_1 \).

**Solution:** Let \( y_0(t) \) be as in Part (b), and let \( y_1(t) \) be as in Part (c). Set \( y(t) = a_0 y_0(t) + a_1 y_0(t) \). Then
\[ y''(t) - t^2 y(t) = a_0 y_0''(t) + a_1 y_1''(t) - t(a_0 y_0(t) + a_1 y_0(t)) \]
\[ = a_0 (y_0''(t) - t^2 y_0(t)) + a_1 (y_1''(t) - t^2 y_1(t)) \]
\[ = a_0 \cdot 0 + a_1 \cdot 0 = 0. \]
So \( t \mapsto y(t) \) is a solution to (1). It is immediate to check that \( y(0) = a_0 \) and \( y'(0) = a_1 \).

Therefore the series solution is
\[ y(t) = \sum_{n=0}^{\infty} c_n t^n, \]
with \( c_0 = a_0 \),
\[ c_{4n} = \frac{a_0}{3 \cdot 4 \cdot 7 \cdot 8 \cdots (4n - 1) \cdot (4n)} \]
for \( n = 1, 2, 3, 4, \ldots \), \( c_1 = a_1 \),
\[ c_{4n+1} = \frac{a_1}{4 \cdot 5 \cdot 8 \cdots (4n) \cdot (4n + 1)} \]
for \( n = 1, 2, 3, 4, \ldots \), and \( c_{4n+2} = 0 \) for \( n = 0, 1, 2, 3, \ldots \).

(e) Find the interval of convergence of the series in Part (d).

**Solution:** We will show that the series in Part (b) for \( y_0(t) \) and the series in Part (c) for \( y_1(t) \) both converge for all real numbers \( t \). It will follow that the series in Part (d) converges for all real numbers \( t \). For the both series in Part (b) and Part (c), we use the Ratio Test.

In Part (c), for \( t \neq 0 \), we have
\[ \lim_{n \to \infty} \left| \frac{t^{4n+1}}{3 \cdot 4 \cdot 7 \cdot 8 \cdots (4n - 1) \cdot (4n) \cdot (4n + 1)} \right| = \lim_{n \to \infty} \frac{|t|^4}{(4n + 3)(4n + 4)} = 0. \]
Since \( 0 < 1 \), the series is absolutely convergent for every \( t \neq 0 \) (and, of course, for \( t = 0 \)). So the interval of convergence is \((-\infty, \infty)\).
In Part (c), for \( t \neq 0 \), we have
\[
\lim_{n \to \infty} \frac{4(n+1)/4^{n+1} + 5/4^{n+1}}{4(n+1)/4^{n+1} - 4(n+1)/4^{n+1} + 1} = \lim_{n \to \infty} \frac{|t|^4}{4n+4(4n+5)} = 0.
\]
Since \( 0 < 1 \), the series is absolutely convergent for every \( t \neq 0 \) (and, of course, for \( t = 0 \)). So the interval of convergence is \((-\infty, \infty)\).

17. (6 points.) For the differential equation \( y''(x) + xy'(x) + 2y(x) = 0 \), find the recurrence relation for the coefficients of the power series solutions centered at 0, and find two independent power series solutions through the terms of degree 3. Be sure to show your reasoning in mathematically and notationally correct steps.

**Solution:** We assume that the solution \( y(x) \) is given by
\[
y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots.
\]
Then
\[
y'(x) = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + \cdots
\]
and
\[
y''(x) = 2c_2 + 3 \cdot 2c_3 x + 4 \cdot 3c_4 x^2 + 5 \cdot 4c_5 x^3 + 6 \cdot 5c_6 x^4 + \cdots.
\]
Therefore (showing, as it turns out, more terms than are actually needed)
\[
y''(x) + xy'(x) + 2y(x)
= 2c_2 + 3 \cdot 2c_3 x + 4 \cdot 3c_4 x^2 + 5 \cdot 4c_5 x^3 + 6 \cdot 5c_6 x^4 + \cdots
+ x(c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + \cdots)
+ 2(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots)
= 2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + 30c_6 x^4 + \cdots
+ c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + \cdots
+ 2c_0 + 2c_1 x + 2c_2 x^2 + 2c_3 x^3 + 2c_4 x^4 + \cdots
= (2c_2 + 2c_0) + (6c_3 + 3c_1) x + (12c_4 + 4c_2) x^2
+ (20c_5 + 5c_3) x^3 + (30c_6 + 6c_4) x^4 + \cdots.
\]
The coefficient of \( x^n \) is \((n+1)(n+2)c_{n+2} + (n+2)c_n\).

This series is supposed to converge to the zero function, so, by uniqueness of power series expansions, all the coefficients are zero. That is, \((n+1)(n+2)c_{n+2} + (n+2)c_n = 0\). Since \(n+1 \neq 0\) and \(n+2 \neq 0\) (as \(n \geq 0\)), the recurrence relation is just
\[
c_{n+2} = -\frac{c_n}{n+1}.
\]
For one of our two independent solutions, we choose the initial conditions \(y_0(0) = 1\) and \(y_0'(0) = 0\), and for the other we choose the initial conditions \(y_1(0) = 0\) and \(y_1'(0) = 1\). For the first, the initial conditions give
\[
c_0 = 1 \quad \text{and} \quad c_1 = 0.
\]
Applying (2) twice gives
\[
c_2 = -\frac{c_0}{0+1} = -1 \quad \text{and} \quad c_3 = -\frac{c_1}{1+1} = 0.
\]
Therefore
\[
y_0(x) = 1 - x^2 + \cdots.
\]
Similarly, for \(y_1\) the initial conditions give
\[
c_0 = 0 \quad \text{and} \quad c_1 = 1.
\]
Applying (2) twice gives
\[ c_2 = -\frac{c_0}{0+1} = 0 \quad \text{and} \quad c_3 = -\frac{c_1}{1+1} = -\frac{1}{2}. \]
Therefore
\[ y_1(x) = x - \frac{x^3}{2} + \cdots. \]

18. (6 points.) For the differential equation \((2 + x^2)y''(x) - xy'(x) + 4y(x) = 0\), find the recurrence relation for the coefficients of the power series solutions centered at 0, and find two independent power series solutions through the terms of degree 3. Be sure to show your reasoning in mathematically and notationally correct steps.

**Solution:** We assume that the solution \(y(x)\) is given by
\[ y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots. \]
Then
\[ y'(x) = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + \cdots \]
and
\[ y''(x) = 2c_2 + 3 \cdot 2c_3 x + 4 \cdot 3c_4 x^2 + 5 \cdot 4c_5 x^3 + 6 \cdot 5c_6 x^4 + \cdots. \]
Therefore (showing, as it turns out, more terms than are actually needed)
\[
(2 + x^2)y''(x) - xy'(x) + 4y(x)
= 2(2c_2 + 3 \cdot 2c_3 x + 4 \cdot 3c_4 x^2 + 5 \cdot 4c_5 x^3 + 6 \cdot 5c_6 x^4 + \cdots)
+ x^2(2c_2 + 3 \cdot 2c_3 x + 4 \cdot 3c_4 x^2 + 5 \cdot 4c_5 x^3 + 6 \cdot 5c_6 x^4 + \cdots)
- x(c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + \cdots)
+ 4(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots)
= 4c_2 + 12c_3 x + 24c_4 x^2 + 40c_5 x^3 + 60c_6 x^4 + \cdots
+ 2c_2 x^2 + 3 \cdot 2c_3 x^3 + 4 \cdot 3c_4 x^4 + 5 \cdot 4c_5 x^5 + 6 \cdot 5c_6 x^6 + \cdots
- c_1 x - 2c_2 x^2 - 3c_3 x^3 - 4c_4 x^4 - 5c_5 x^5 - \cdots
+ 4c_0 + 4c_1 x + 4c_2 x^2 + 4c_3 x^3 + 4c_4 x^4 + \cdots
= (4c_2 + 4c_0) + (18c_3 - 3c_1) x + (24c_4 + 4c_2) x^2
+ (40c_5 + 7c_3) x^3 + (60c_6 + 12c_4) x^4 + \cdots.
\]
The coefficient of \(x^n\) is
\[(n + 1)(n + 2)c_{n+2} + (n - 1)nc_n - nc_n + 4c_n = (n + 1)(n + 2)c_{n+2} + (n^2 - 2n + 4)c_n = 0. \]
This series is supposed to converge to the zero function, so, by uniqueness of power series expansions, all the coefficients are zero. That is, \((n + 1)(n + 2)c_{n+2} + (n^2 - 2n + 4)c_n = 0\). Since \(n + 1 \neq 0\) and \(n + 2 \neq 0\) (as \(n \geq 0\), the recurrence relation is
\[
(3) \quad c_{n+2} = -\frac{(n^2 - 2n + 4)c_n}{(n+1)(n+2)}. \]
For one of our two independent solutions, we choose the initial conditions \(y_0(0) = 1\) and \(y_0'(0) = 0\), and for the other we choose the initial conditions \(y_1(0) = 0\) and \(y_1'(0) = 1\). For the first, the initial conditions give
\[ c_0 = 1 \quad \text{and} \quad c_1 = 0. \]
Applying (3) twice gives
\[ c_2 = -\frac{(0^2 - 2 \cdot 0 + 4)c_0}{(0+1)(0+2)} = -2 \quad \text{and} \quad c_3 = -\frac{(1^2 - 2 \cdot 1 + 4)c_1}{(1+1)(1+2)} = 0. \]
Therefore
\[ y_0(x) = 1 - 2x^2 + \cdots. \]
Similarly, for $y_1$ the initial conditions give
\[ c_0 = 0 \quad \text{and} \quad c_1 = 1. \]

Applying (3) twice gives
\[ c_2 = \frac{(0^2 - 2 \cdot 0 + 4)c_0}{(0 + 1)(0 + 2)} = 0 \quad \text{and} \quad c_3 = \frac{(1^2 - 2 \cdot 1 + 4)c_1}{(1 + 1)(1 + 2)} = -\frac{1}{2}. \]

Therefore
\[ y_1(x) = x - \frac{x^3}{2} + \cdots. \]

19. (5 points.) Suppose the function $x \mapsto y(x)$ is a solution to the differential equation $y''(x) + xy'(x) + y(x) = 0$, and satisfies $y(0) = 1$ and $y'(0) = 0$. Find $y''(0)$, $y'''(0)$, and $y^{(4)}(0)$. Be sure to show your reasoning in mathematically and notationally correct steps.

\textit{Solution:} Put $x = 0$ in $y''(x) + xy'(x) + y(x) = 0$ getting $y''(0) + y(0) = 0$. Combine with $y(0) = 1$ to get $y''(0) = -1$.

Next, differentiate both sides of the equation $y''(x) + xy'(x) + y(x) = 0$ with respect to $x$, getting
\[ y'''(x) + y'(x) + xy''(x) + y'(x) = 0, \]
that is,
\[ y'''(x) + xy''(x) + 2y'(x) = 0. \]

Put $x = 0$ in this equation, getting $y'''(0) + 2y'(0) = 0$. Combine with $y'(0) = 0$ to get $y'''(0) = 0$.

Finally, differentiate both sides of the equation (4) with respect to $x$, getting
\[ y''''(x) + y''(x) + xy'''(x) + 2y''(x) = 0, \]
that is,
\[ y''''(x) + xy'''(x) + 3y''(x) = 0. \]

Put $x = 0$ in this equation, getting $y''''(0) + 3y''(0) = 0$. Combine with $y''(0) = -1$ to get $y''''(0) = 3$.

20. (5 points.) Suppose the function $x \mapsto y(x)$ is a solution to the differential equation $y''(x) + \sin(x)y'(x) + \cos(x)y(x) = 0$, and satisfies $y(0) = 0$ and $y'(0) = 1$. Find $y''(0)$, $y'''(0)$, and $y^{(4)}(0)$. Be sure to show your reasoning in mathematically and notationally correct steps.

\textit{Solution:} Put $x = 0$ in $y''(x) + \sin(x)y'(x) + \cos(x)y(x) = 0$ getting $y''(0) + \sin(0)y'(0) + \cos(0)y(0) = 0$, that is, $y''(0) + y(0) = 0$. Combine with $y(0) = 0$ to get $y''(0) = 0$.

Next, differentiate both sides of the equation $y''(x) + \sin(x)y'(x) + \cos(x)y(x) = 0$ with respect to $x$, getting
\[ y'''(x) + \cos(x)y'(x) + \sin(x)y''(x) - \sin(x)y(x) + \cos(x)y'(x) = 0, \]
that is,
\[ y'''(x) + \sin(x)y''(x) + 2\cos(x)y'(x) - \sin(x)y(x) = 0. \]

Put $x = 0$ in this equation, getting
\[ y'''(0) + \sin(0)y''(0) + 2\cos(0)y'(0) - \sin(0)y(0) = 0, \]
that is, $y'''(0) + 2y'(0) = 0$. Combine with $y'(0) = 1$ to get $y'''(0) = -2$.

Finally, differentiate both sides of the equation (5) with respect to $x$, getting
\[ y''''(x) + \cos(x)y'(x) + \sin(x)y''(x) - 2\sin(x)y'(x) + 2\cos(x)y''(x) - \cos(x)y(x) - \sin(x)y'(x) = 0, \]
that is,
\[ y''''(x) + \sin(x)y''(x) + 3\cos(x)y'(x) - 3\sin(x)y'(x) - \cos(x)y(x) = 0. \]

Put $x = 0$ in this equation, getting $y''''(0) + 3y''(0) - y(0) = 0$. Combine with $y''(0) = 0$ and $y(0) = 0$ to get $y''''(0) = 0$. 