WORKSHEET SOLUTION: POWER SERIES SOLUTION TO A DIFFERENTIAL EQUATION

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1. Find the terms through degree 4 of the general power series solution to the equation \( y'(x) = (x + 2)y(x) + 1 \) centered at 0. (The next problem assumes you called the coefficients in your series \( a_n \) for \( n = 0, 1, 2, \ldots \). Thus, \( y(x) = \sum_{n=0}^{\infty} a_n x^n \).
Start by letting \( a_0 = y(0) \) be arbitrary. Then find \( a_1 \) in terms of \( a_0 \), etc.

**Solution:** Assume \( y(x) \) has the form \( y(x) = a_0 + a_1 x + a_2 x^2 + \cdots \). Then \( y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots \). Substituting in \( y'(x) = (x + 2)y(x) + 1 \) gives

\[
a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = x(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) + 2(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) + 1
\]

\[
= (1 + 2a_0) + (a_0 + 2a_1)x + (a_1 + 2a_2)x^2 + (a_2 + 2a_3)x^3 + (a_3 + 2a_4)x^4 + \cdots.
\]

So we get

\[
a_1 = 1 + 2a_0, \quad a_2 = \frac{1}{2}(a_0 + 2a_1) = \frac{1}{2}(a_0 + 2(1 + 2a_0)) = \frac{1}{2}(5a_0 + 2),
\]

\[
a_3 = \frac{1}{3}(a_1 + 2a_2) = \frac{1}{3}(1 + 2a_0 + (5a_0 + 2)) = \frac{1}{3}(7a_0 + 3),
\]

and

\[
a_4 = \frac{1}{4}(a_2 + 2a_3) = \frac{1}{4}\left(\frac{1}{2}(5a_0 + 2) + \frac{2}{3}(7a_0 + 3)\right) = \frac{1}{24}(43a_0 + 18).
\]

Therefore

\[
y(x) = a_0 + (1 + 2a_0)x + \frac{1}{2}(5a_0 + 2)x^2 + \frac{1}{3}(7a_0 + 3)x^3 + \frac{1}{24}(43a_0 + 18)x^4 + \cdots.
\]

2. For \( n = 1, 2, 3, 4, \ldots \), give an equation for \( a_{n+1} \) in terms of \( a_0, a_1, \ldots, a_n \).
(This is a “recursive equation” for the coefficients. In this case, you only need \( a_n \) and \( a_{n-1} \) on the right.)

**Solution:** For \( n = 1, 2, 3, 4, \ldots \), the coefficient of \( x^n \) in \( y'(x) \) is \((n + 1)a_{n+1}\) and the coefficient of \( x^n \) in \((x+2)y(x) + 1\) is \( a_{n-1} + 2a_n \). These must be equal, so the equation is

\[
a_{n+1} = \frac{1}{n+1}(a_{n-1} + 2a_n).
\]

Note: \( n = 0 \) is a special case.
Note: You used the cases \( n = 1, 2, 3 \) in Problem 1.

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