1. (1 point) Are you awake?

Solution: I hope you were!

2. (9 points.) Determine whether or not the series \( \sum_{n=1}^{\infty} e^{-1/n^2} \) is convergent. Be sure to show your reasoning in mathematically and notationally correct steps.

Solution: We have \( \lim_{n \to \infty} \left( -\frac{1}{n^2} \right) = 0 \). Since \( x \mapsto e^x \) is continuous at 0, it follows that \( \lim_{n \to \infty} e^{-1/n^2} = e^0 = 1 \). Since the summands don’t converge to 0, the series \( \sum_{n=1}^{\infty} (-1)^n e^{-1/n^2} \) diverges.

It is a notational error to write “\( \lim_{n \to \infty} -\frac{1}{n^2} \)” without parentheses. Since the order of operations specifies that addition and subtraction are done before taking limits, the crossed out expression means to take the limit of some unspecified thing, and afterwards subtract \( \frac{1}{n^2} \).

3. (9 points.) Define a sequence \( (a_n)_{n=1}^{\infty} \) by \( a_n = \ln(n^2 + 11) \) for \( n = 1, 2, 3, 4, \ldots \). Find its limit (possibly \( \infty \) or \( -\infty \)), or explain why the sequence diverges but not to \( \infty \) or \( -\infty \). Give reasons in mathematically and notationally correct steps.

Solution: We have \( \lim_{n \to \infty} \ln(x) = \infty \), and \( n^2 + 11 \to \infty \) as \( n \to \infty \), so \( \lim_{n \to \infty} \ln(n^2 + 11) = \infty \). Thus the sequence \( (a_n)_{n=1}^{\infty} \) diverges to \( \infty \).

4. (10 points.) Determine whether or not the series \( \sum_{n=0}^{\infty} \frac{3^{n+7}}{2^{2n}} \) is convergent. If the series is convergent, find its sum. Be sure to show your reasoning in mathematically and notationally correct steps.

Solution: Rewrite

\[
\sum_{n=0}^{\infty} \frac{3^{n+7}}{2^{2n}} = \sum_{n=0}^{\infty} \frac{3^{n+7}}{4^n} = \sum_{n=0}^{\infty} 3^7 \left( \frac{3}{4} \right)^n.
\]
This is a geometric series whose common ratio is $\frac{3}{4}$. Since $-1 < \frac{3}{4} < 1$, the series converges. Moreover,

$$\sum_{n=0}^{\infty} \frac{3^{n+7}}{2^{2n}} = 3^7 \sum_{n=0}^{\infty} \left( \frac{3}{4} \right)^n = 3^7 \left( \frac{1}{1 - \frac{3}{4}} \right) = 3^7 \cdot 4.$$

There is no need to multiply out $3^7$ or $3^7 \cdot 4$.

5. (10 points.) Use the Integral Test to determine whether the series $\sum_{n=1}^{\infty} \frac{n}{7 + n^2}$ is convergent. Be sure to show your reasoning, in particular checking the appropriate hypotheses, in mathematically and notationally correct steps.

Solution: We take $f(x) = \frac{x}{7 + x^2}$. We need to check that $f(x) \geq 0$ beyond some point; this is clearly true for all $x$ in $(0, \infty)$. We need to check that $f$ is nonincreasing beyond some point. For this, we look at

$$f'(x) = \frac{7 + x^2 - x \cdot 2x}{(7 + x^2)^2} = \frac{7 - x^2}{(7 + x^2)^2}.$$ 

This expression is negative when $x \geq 3$ (in fact, when $x > \sqrt{7}$). Therefore $f$ is nonincreasing on $(3, \infty)$.

(A correct solution must show that you checked the hypotheses for the Integral Test.)

Now

$$\int_{3}^{\infty} \frac{x}{7 + x^2} \, dx = \left. \frac{1}{2} \ln(7 + x^2) \right|_{1}^{\infty} = \lim_{b \to \infty} \left( \frac{1}{2} \ln(7 + b^2) \left|_{1}^{b} \right. - \ln(7 + 3^2) \right) = \infty.$$ 

So the improper integral diverges. Therefore $\sum_{n=1}^{\infty} \frac{n}{7 + n^2}$ diverges.

6. (10 points.) Find the Taylor polynomial for the function $g(x) = x \ln(x)$ of degree 4 centered at $x = 3$ (not $x = 0$). (Simplify but don’t multiply out the coefficients.)

Solution: We use the formula

$$T_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

for the $n$-th Taylor polynomial, with $n = 4$ and $a = 3$. The required derivatives are:

$$f(x) = x \ln(x), \quad \text{so} \quad f(3) = 3 \ln(3),$$

$$f'(x) = \ln(x) + 1, \quad \text{so} \quad f'(3) = \ln(3) + 1,$$

$$f''(x) = \frac{1}{x}, \quad \text{so} \quad f''(3) = \frac{1}{3},$$

$$f'''(x) = -\frac{1}{x^2}, \quad \text{so} \quad f'''(3) = -\frac{1}{3^2},$$

and so on.
and 
\[ f^{(4)}(x) = \frac{2}{x^3}, \quad \text{so} \quad f^{(4)}(3) = \frac{2}{3^3}. \]

Therefore the required polynomial is
\[
T_n(x) = 3 \ln(3) + [\ln(3) + 1](x - 3) + \left( \frac{1}{2!} \right) \left( \frac{1}{3} \right) (x - 3)^2 \\
- \left( \frac{1}{3!} \right) \left( \frac{1}{3^2} \right) (x - 3)^3 + \left( \frac{1}{4!} \right) \left( \frac{2}{3^3} \right) (x - 3)^4
\]
\[
= 3 \ln(3) + [\ln(3) + 1](x - 3) + \frac{1}{6}(x - 3)^2 - \frac{1}{3! \cdot 3^2}(x - 3)^3 + \frac{1}{4 \cdot 3 \cdot 3^3}(x - 3)^4.
\]

Alternate solution (outline): Find the Taylor polynomial for the function \( f(x) = \ln(x) \) of degree 4 centered at \( x = 3 \), multiply it by \((x - 3) + 3\), and sort terms appropriately.

7. (6 points.) Let \( (a_n)_{n=1}^{\infty} \) be a sequence, and let \( L \) be a real number. State the precise definition of what it means to have \( \lim_{n \to \infty} a_n = L \).

Solution: For every \( \varepsilon > 0 \) there is a positive integer \( N \) such that for every integer \( n \geq N \), we have \( |a_n - L| < \varepsilon \).

Alternate solution: For every \( \varepsilon > 0 \) there is a positive integer \( N \) such that for every integer \( n > N \), we have \( |a_n - L| < \varepsilon \).

The two versions are equivalent, because one can choose \( N \) to be larger in the first version. Our current book uses the first version, but I will take either.

8. (6 points.) Define a sequence \( (y_n)_{n=1}^{\infty} \) by \( y_n = \frac{4n^2 + 5}{n^2 + 1} \) for \( n = 1, 2, 3, \ldots \). For \( \varepsilon = 0.02 \), find some integer \( N > 0 \) such that for all \( n \geq N \) we have \( |y_n - 4| < \varepsilon \), and show in mathematically and notationally correct steps that your choice works. (You need not find the best value of \( N \).)

Scratchwork (not needed as part of the solution): For \( n = 1, 2, 3, \ldots \), we have
\[
y_n - 4 = \frac{4n^2 + 5}{n^2 + 1} - 4 = \frac{4n^2 + 5 - 4(n^2 + 1)}{n^2 + 1} = \frac{1}{n^2 + 1}.
\]

So
\[
|y_n - 4| = \left| \frac{1}{n^2 + 1} \right| = \frac{1}{n^2 + 1}.
\]

We want
\[
\frac{1}{n^2 + 1} < 0.02 = \frac{1}{50}.
\]
This happens if \( n^2 + 1 > 50 \), so \( n = 10 \) will clearly work.
Solution: We take $N = 10$. Consider an arbitrary integer $n$ such that $n \geq 10$. Then $n^2 \geq 100$, so

$$|y_n - 4| = \left| \frac{4n^2 + 5}{n^2 + 1} - 4 \right| = \frac{1}{n^2 + 1} < \frac{1}{50} = 0.02.$$ 

That is, $|y_n - 4| < 0.02$. So $N = 10$ works.

(This is not the best choice of $N$; in fact, $N = 7$ will work, but $N = 6$ won’t.)

9. (9 points.) Define $f(x) = \cos(5x^3)$ for all real $x$. Find $f^{(12)}(0)$. (Remember to show your work in mathematically and notationally correct steps. Simplify your answer but don’t multiply out powers, factorials, etc.)

Solution: The Taylor series for $\cos(x)$ is

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots .$$

Putting $5x^3$ in place of $x$ and discarding terms of degree greater than 12, we get the following for the Taylor polynomial of degree 12 for $\cos(5x^3)$ centered at 0:

$$1 - \frac{(5x^3)^2}{2!} + \frac{(5x^3)^4}{4!} = 1 - \frac{5^2}{2!}x^6 + \frac{3^4}{4!}x^{12}.$$

Select the coefficient of $x^{12}$ and multiply it by $12!$, getting

$$f^{(12)}(0) = 12! \left( \frac{3^4}{4!} \right) = \frac{3^4 \cdot 12!}{4!}.$$

10. (10 points.) Define a sequence $(a_n)_{n=1}^{\infty}$ by $a_n = \frac{\ln(n) + 12}{12 \ln(n) + 7}$ for $n = 1, 2, 3, 4, \ldots$. Find its limit (possibly $\infty$ or $-\infty$), or explain why the sequence diverges but not to $\infty$ or $-\infty$. Give reasons in mathematically and notationally correct steps.

Solution: For $n = 2, 3, 4, \ldots$ (but not for $n = 1$), we have

$$\lim_{n \to \infty} \frac{\ln(n) + 12}{12 \ln(n) + 7} = \lim_{n \to \infty} \frac{\frac{\ln(n)}{1!} (\ln(n) + 12)}{12 \frac{\ln(n) + 7}{7}} = \lim_{n \to \infty} \frac{1 + 12 \frac{\ln(n)}{1!}}{12 + 7 \frac{\ln(n)}{1!}} = \frac{1 + 12 \cdot 0}{12 + 7 \cdot 0} = \frac{1}{12}.$$

Alternate solution: We use L’Hopital’s Rule to find $\lim_{x \to \infty} \frac{\ln(x) + 12}{12 \ln(x) + 7}$. Since

$$\lim_{x \to \infty} (\ln(x) + 12) = \infty \quad \text{and} \quad \lim_{x \to \infty} (12 \ln(x) + 7) = \infty,$$
it is legitimate to try L'Hopital's Rule. (You must show that you know this.) Thus,
\[
\lim_{x \to \infty} \frac{\ln(x) + 12}{12 \ln(x) + 7} = \lim_{x \to \infty} \frac{\left(\frac{1}{x}\right)}{\frac{12}{x}} = \lim_{x \to \infty} \frac{1}{12} = \frac{1}{12}.
\]

Therefore
\[
\lim_{n \to \infty} \frac{\ln(n) + 12}{12 \ln(n) + 7} = \frac{1}{12}.
\]

11. (10 points.) Find the Taylor polynomial for the function \( f(x) = \frac{1}{(1 + x)^{1/5}} \) of degree 4 centered at \( x = 0 \). (Simplify but don't multiply out the coefficients.)

Solution: The function is \( f(x) = (1 + x)^{-1/5} \). Use the binomial series, getting
\[
1 + \left(-\frac{1}{5}\right)x + \frac{1}{2!}\left(-\frac{1}{5}\right)\left(-\frac{6}{5}\right)x^2 + \frac{1}{3!}\left(-\frac{1}{5}\right)\left(-\frac{6}{5}\right)\left(-\frac{11}{5}\right)x^3
\]
\[
+ \frac{1}{4!}\left(-\frac{1}{5}\right)\left(-\frac{6}{5}\right)\left(-\frac{11}{5}\right)\left(-\frac{16}{5}\right)x^4
\]
\[
= 1 - \left(\frac{1}{5}\right)x + \left(\frac{1}{2 \cdot 5^2}\right)x^2 - \left(\frac{1}{3! \cdot 5^3}\right)x^3 + \left(\frac{1}{4! \cdot 5^4}\right)x^4.
\]

Alternate solution: The requested degree of Taylor polynomial is small enough that it is reasonable to do this problem directly.

We use the formula
\[
T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n
\]
for the \(n\)-th Taylor polynomial, with \( n = 4 \) and \( a = 0 \). The required derivatives are:

\( f(x) = (1 + x)^{-1/5} \), so \( f(0) = 1 \),

\( f'(x) = -\frac{1}{5} \left(1 + x\right)^{-6/5} \), so \( f'(0) = -\frac{1}{5} \),

\( f''(x) = \left(-\frac{6}{5}\right)\left(-\frac{1}{5}\right) \left(1 + x\right)^{-11/5} \), so \( f''(0) = \left(-\frac{6}{5}\right)\left(-\frac{1}{5}\right) = \frac{6 \cdot 1}{5^2} \),

\( f'''(x) = \left(-\frac{11}{5}\right)\left(-\frac{6}{5}\right)\left(-\frac{1}{5}\right) \left(1 + x\right)^{-16/5} \), so \( f'''(0) = \left(-\frac{11}{5}\right)\left(-\frac{6}{5}\right)\left(-\frac{1}{5}\right) = \frac{11 \cdot 6 \cdot 1}{5^3} \),

and

\( f^{(4)}(x) = \left(-\frac{16}{5}\right)\left(-\frac{11}{5}\right)\left(-\frac{6}{5}\right)\left(-\frac{1}{5}\right) \left(1 + x\right)^{-21/5} \),

so

\( f^{(4)}(0) = \left(-\frac{16}{5}\right)\left(-\frac{11}{5}\right)\left(-\frac{6}{5}\right)\left(-\frac{1}{5}\right) = \frac{16 \cdot 11 \cdot 6 \cdot 1}{5^4} \).

Therefore the required polynomial is
\[
T_4(x) = 1 + \left(-\frac{1}{5}\right)x + \left(\frac{6 \cdot 1}{2! \cdot 5^2}\right)x^2 - \left(\frac{11 \cdot 6 \cdot 1}{3! \cdot 5^3}\right)x^3 + \left(\frac{16 \cdot 11 \cdot 6 \cdot 1}{4! \cdot 5^4}\right)x^4.
\]
12. (10 points.) The sequence \((c_n)_{n=1}^\infty\) is given by the formula
\[ c_n = -11 + \frac{5}{3n^2 + 2} \]
for \(n = 1, 2, 3, \ldots\). Determine whether this sequence is nondecreasing ("increasing" in the textbook), nonincreasing ("decreasing" in the textbook), or not monotone. Also determine whether it is bounded. Remember to show work in a mathematically and notationally correct manner.

**Solution:** The sequence \((c_n)_{n=1}^\infty\) fails to be nondecreasing since
\[ c_2 = -11 + \frac{5}{3 \cdot 2^2 + 2} = -11 + \frac{5}{13} < -10 = -11 + \frac{5}{3 \cdot 1^2 + 2} = c_1. \]
This sequence is nonincreasing, since for \(n = 1, 2, 3, \ldots\) we have \(0 < 3n^2 + 2 < 3(n + 1) + 2\), so
\[ \frac{5}{3n^2 + 2} > \frac{5}{3(n+1)^2 + 2}, \]
whence
\[ c_{n+1} = -11 + \frac{5}{3(n + 1)^2 + 2} \leq -11 + \frac{5}{3n^2 + 2} = c_n. \]
Since the sequence nonincreasing, it is monotone. For \(n = 1, 2, 3, \ldots\), we have
\[ 0 < \frac{5}{3n^2 + 2} \leq \frac{5}{2}, \]
so
\[ -11 < c_n = -11 + \frac{5}{3n^2 + 2} \leq -11 + \frac{5}{2}. \]
Therefore \((c_n)_{n=1}^\infty\) is bounded both above and below. Thus \((c_n)_{n=1}^\infty\) is bounded.

**Alternate solution for boundedness:** We have \(\lim_{n \to \infty} (3n^2 + 2) = \infty\), so
\[ \lim_{n \to \infty} c_n = \lim_{n \to \infty} \left(-11 + \frac{5}{3n^2 + 2}\right) = -11 + 0 = -11. \]
Since convergent sequences are bounded, we conclude that \((c_n)_{n=1}^\infty\) is bounded.

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Extra credit. (15 extra credit points.) Prove directly from the definition of the limit that
\[ \lim_{n \to \infty} \frac{\sqrt[4]{n}}{7\sqrt{n} - 2} = \frac{1}{7}. \]

**Scratchwork (not needed as part of the solution):** Check that
\[ \frac{\sqrt[4]{n}}{7\sqrt{n} - 2} - \frac{1}{7} = \frac{2}{49\sqrt[4]{n} - 14}. \]
So
\[ \left| \frac{\sqrt[4]{n}}{7\sqrt{n} - 2} - \frac{1}{7} \right| = \frac{2}{49\sqrt[4]{n} - 14}. \]
Let \( \varepsilon > 0 \). We want

\[
\frac{2}{49 \sqrt{n} - 14} < \varepsilon.
\]

This says that

\[
49 \sqrt{n} - 14 > \frac{2}{\varepsilon},
\]

or

\[
\sqrt{n} > \frac{1}{49} \left( \frac{2}{\varepsilon} + 14 \right).
\]

This happens if

\[
n > \left( \frac{1}{49} \left( \frac{2}{\varepsilon} + 14 \right) \right)^4.
\]

**Solution:** Let \( \varepsilon > 0 \) be arbitrary. Choose some positive integer \( N \) such that

\[
N > \left( \frac{1}{49} \left( \frac{2}{\varepsilon} + 14 \right) \right)^4.
\]

Now let \( n \) be any positive integer which satisfies \( n \geq N \). Then

\[
\left| \frac{\sqrt{n}}{7 \sqrt{n} - 2} - \frac{1}{7} \right| = \left| \frac{2}{49 \sqrt{n} - 14} \right| = \frac{2}{49 \sqrt{n} - 14} \leq \frac{2}{49 \sqrt{N} - 14} < \frac{2}{49 \left( \frac{1}{49} \left( \frac{2}{\varepsilon} + 14 \right) \right) - 14} = \varepsilon,
\]

as required.