1. (8 points.) Define a sequence \( (a_n)_{n=1}^{\infty} \) by \( a_n = \frac{11e^{2n} - 1}{e^{2n} + 11} \) for strictly positive integers \( n \). Find its limit (possibly \( \infty \) or \( -\infty \)), giving reasons, or explain why the sequence neither converges nor diverges to \( \infty \) or \( -\infty \). Reasons must be given in mathematically and notationally correct form.

**Solution:** We have

\[
\lim_{n \to \infty} \frac{11e^{2n} - 1}{e^{2n} + 11} = \lim_{n \to \infty} \frac{e^{2n}(11e^{2n} - 1)}{e^{2n}(e^{2n} + 11)} = \lim_{n \to \infty} \frac{11 - e^{-2n}}{1 + 11e^{-2n}} = \frac{11 - 0}{1 + 11 \cdot 0} = 11.
\]

2. (7 points.) Let \( (a_n)_{n=1}^{\infty} \) be a sequence, and let \( L \) be a real number. State the precise definition of what it means to have \( \lim_{n \to \infty} a_n = L \).

**Solution:** For every \( \varepsilon > 0 \) there is a positive integer \( N \) such that for every integer \( n > N \), we have \( |a_n - L| < \varepsilon \).

3. (8 points.) Define a sequence \( (x_n)_{n=1}^{\infty} \) by \( x_n = \frac{n}{2n + 1} \) for \( n = 1, 2, \ldots \). For \( \varepsilon = 0.02 \), find some integer \( N > 0 \) such that for all \( n > N \) we have \( |x_n - \frac{1}{2}| < \varepsilon \), and show in mathematically and notationally correct steps that your choice works. (You need not find the best value of \( N \).)

**Scratchwork (not needed as part of the solution):** For \( n = 1, 2, 3, \ldots \) rewrite

\[
\left| x_n - \frac{1}{2} \right| = \left| \frac{n}{2n + 1} - \frac{1}{2} \right| = \left| \frac{2n - (2n + 1)}{2(2n + 1)} \right| = \left| \frac{-1}{2(2n + 1)} \right| = \frac{1}{2(2n + 1)}.
\]

We want to have

\[
\frac{1}{2(2n + 1)} < 0.02,
\]

so

\[
2n + 1 > \frac{1}{(2)(0.02)} = 25,
\]

so we want \( n > 12 \). So it is enough to take \( N = 12 \).

**Solution:** Take \( N = 20 \). [This is not the best possible choice; the scratchwork above shows that \( N = 12 \) is the best possible choice.] Let \( n \) be any integer with \( n > N \). Then

\[
\left| x_n - \frac{1}{2} \right| = \left| \frac{n}{2n + 1} - \frac{1}{2} \right| = \left| \frac{-1}{2(2n + 1)} \right| = \frac{1}{2(2n + 1)} < \frac{1}{2(2N + 1)} = \frac{1}{2(2 \times 20 + 1)} = \frac{1}{82} < \frac{1}{50} = 0.02.
\]
4. (8 points.) Determine whether or not the series $\sum_{n=1}^{\infty} (-1)^n \sin \left( \frac{\pi}{2} - \frac{1}{2n} \right)$ is convergent. Be sure to show your reasoning. If the series is convergent, find its sum. Use mathematically and notationally correct steps.

**Solution:** We have

$$\lim_{n \to \infty} \left( \frac{\pi}{2} - \frac{1}{2n} \right) = \frac{\pi}{2}.$$ 

Since $x \mapsto \sin(x)$ is continuous at $\frac{\pi}{2}$, it follows that

$$\lim_{n \to \infty} \sin \left( \frac{\pi}{2} - \frac{1}{2n} \right) = \sin \left( \frac{\pi}{2} \right) = 1.$$ 

Therefore

$$\lim_{n \to \infty} (-1)^n \sin \left( \frac{\pi}{2} - \frac{1}{2n} \right)$$

does not exist. Since the summands don’t converge to 0, the series $\sum_{n=1}^{\infty} (-1)^n \sin \left( \frac{\pi}{2} - \frac{1}{2n} \right)$ is divergent.

5. (8 points.) Determine whether or not the series $\sum_{n=0}^{\infty} \frac{4^n}{3^{n+2}}$ is convergent. Be sure to show your reasoning. If the series is convergent, find its sum. Use mathematically and notationally correct steps.

**Solution:** This is a geometric series with common ratio $\frac{4}{3}$. Since $\frac{4}{3} > 1$, the summands don’t approach zero (in fact, $\lim_{n \to \infty} \frac{4^n}{3^{n+2}} = \infty$), so the series is divergent.

6. (8 points.) Determine whether or not the series $\sum_{n=1}^{\infty} [\sqrt{n+1} - \sqrt{n+2}]$ is convergent. Be sure to show your reasoning. If the series is convergent, find its sum. Use mathematically and notationally correct steps.

**Solution:** This is a telescoping series. The $n$th partial sum $s_n$ is

$$s_n = \sum_{k=1}^{n} [\sqrt{k+1} - \sqrt{k+2}] = \sqrt{2} - \sqrt{n+2}.$$ 

Since $\lim_{n \to \infty} \sqrt{n} = \infty$, it follows that $\lim_{n \to \infty} \sqrt{n+2} = \infty$, so

$$\sum_{n=1}^{\infty} [\sqrt{n} - \sqrt{n+1}] = \lim_{n \to \infty} s_n = \lim_{n \to \infty} [\sqrt{2} - \sqrt{n+2}] = -\infty.$$
Thus, the series diverges.

7. (8 points.) Find the Taylor polynomial for the function \( f(x) = e^{x-2} \) of degree 6 centered at \( x = 0 \). (Don’t multiply out the coefficients.) Use mathematically and notationally correct steps.

**Solution:** We use the formula

\[
p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n
\]

for the \( n \)-th Taylor polynomial, with \( n = 6 \). We have \( f'(x) = e^{x-2} \), \( f''(x) = e^{x-2} \), etc.; in fact, \( f^{(n)}(x) = e^{x-2} \) for all positive integers \( n \). So

\[
f(0) = e^{-2}, \quad f'(0) = e^{-2}, \quad f''(0) = e^{-2}, \ldots, \quad f^{(6)}(0) = e^{-2}.
\]

Therefore the degree 6 Taylor polynomial is

\[
e^{-2} + e^{-2}x + \frac{e^{-2}}{2!}x^2 + \frac{e^{-2}}{3!}x^3 + \frac{e^{-2}}{4!}x^4 + \frac{e^{-2}}{5!}x^5 + \frac{e^{-2}}{6!}x^6.
\]

The answer

\[
q(x) = 1 + (x - 2) + \frac{1}{2!}(x - 2)^2 + \cdots + \frac{1}{6!}(x - 2)^6
\]

isn’t right. For example, the next term, \( \frac{1}{7!}(x - 2)^7 \), contributes something of degree zero. Another way to see this is to check that

\[
q(0) = 1 + (-2) + \frac{1}{2!}(-2)^2 + \frac{1}{3!}(-2)^3 + \frac{1}{4!}(-2)^4 + \frac{1}{5!}(-2)^5 + \frac{1}{6!}(-2)^6 \neq e^{-2}.
\]

8. (8 points.) Define \( f(x) = \cos(2x^4) \) for all real \( x \). Find \( f^{(8)}(0) \). (Remember to show your work in mathematically and notationally correct steps. Simplify your answer but don’t multiply out powers, factorials, etc.)

**Solution:** The Taylor series for \( \cos(x) \) is

\[
1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots.
\]

Putting \( 2x^4 \) in place of \( x \) and discarding terms of degree greater than 8 gives

\[
\cos(2x^4) \approx 1 - \frac{(2x^4)^2}{2!} = 1 - 2x^8.
\]

So \( f^{(8)}(0) = -2 \cdot 8! \).
9. (8 points.) The sequence \((c_n)_{n=1}^\infty\) is given by the formula
\[c_n = 3 + \frac{(-1)^{n+1}}{2n + 7}\]
for \(n = 1, 2, 3, \ldots\). Determine whether this sequence is nondecreasing (“increasing” in the textbook), nonincreasing (“decreasing” in the textbook), or not monotone. Also determine whether it is bounded. Remember to show work in a mathematically and notationally correct manner.

**Solution:** The sequence \((c_n)_{n=1}^\infty\) fails to be nonincreasing since
\[c_2 = 3 - \frac{1}{2 \cdot 2 + 7} = 3 - \frac{1}{11} < 3 + \frac{1}{13} = 3 + \frac{1}{2 \cdot 3 + 7} = c_3.\]
This sequence fails to be nonincreasing since
\[c_1 = 3 + \frac{1}{2 \cdot 1 + 7} = 3 + \frac{1}{9} > 3 - \frac{1}{11} = 3 + \frac{1}{2 \cdot 2 + 7} = c_2.\]
Since it is neither nondecreasing nor nonincreasing, it is not monotone.

(A correct proof must give explicit cases both of \(n > m\) such that \(c_n < c_m\) and of \(n > m\) such that \(c_n > c_m\), as was done above. However, many choices other than those above are possible.)

For \(n = 1, 2, 3, \ldots\), we have \(0 < 9 \leq 2n + 7\), so
\[-\frac{1}{9} \leq \frac{(-1)^{n+1}}{2n + 7} \leq \frac{1}{9},\]
whence
\[3 - \frac{1}{9} \leq c_n \leq 3 + \frac{1}{9}.\]
Therefore \((c_n)_{n=1}^\infty\) is bounded both above and below. Thus \((c_n)_{n=1}^\infty\) is bounded.

**Alternate solution for boundedness:** We have \(\lim_{n \to \infty} (2n + 7) = \infty\), so
\[\lim_{n \to \infty} c_n = \lim_{n \to \infty} \left(3 + \frac{(-1)^{n+1}}{2n + 7}\right) = 3 + 0 = 3.\]
Since convergent sequences are bounded, we conclude that \((c_n)_{n=1}^\infty\) is bounded.

10. (8 points.) Find the Taylor polynomial for the function \(f(x) = \sin(x)\) of degree 7 centered at \(x = \pi\) (not \(x = 0\)). (Simplify but don’t multiply out the coefficients.)

**Solution:**
\[T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.\]
for the \(n\)-th Taylor polynomial, with \(n = 7\) and \(a = \frac{\pi}{2}\). The required derivatives are:
\[f(x) = \sin(x), \quad \text{so} \quad f(\pi) = \sin(\pi) = 0,\]
\[f'(x) = \cos(x), \quad \text{so} \quad f'(\pi) = \cos(\pi) = -1,\]
\[f''(x) = -\sin(x), \quad \text{so} \quad f''(\pi) = -\sin(\pi) = 0,\]
\[ f'''(x) = -\cos(x), \quad \text{so} \quad f'''(\pi) = -\cos(\pi) = 1, \]
\[ f^{(4)}(x) = \sin(x), \quad \text{so} \quad f^{(4)}(\pi) = \sin(\pi) = 0, \]
\[ f^{(5)}(x) = \cos(x), \quad \text{so} \quad f^{(5)}(\pi) = \cos(\pi) = -1, \]
\[ f^{(6)}(x) = -\sin(x), \quad \text{so} \quad f^{(6)}(\pi) = -\sin(\pi) = 0, \]
and
\[ f^{(7)}(x) = -\cos(x), \quad \text{so} \quad f^{(7)}(\pi) = -\cos(\pi) = 0. \]

Therefore the required polynomial is
\[ T_n(x) = -(x - \pi) + \frac{1}{3!}(x - \pi)^3 - \frac{1}{5!}(x - \pi)^5 + \frac{1}{7!}(x - \pi)^7. \]

**Alternate solution:** Start with the Taylor polynomial for \( \sin(x) \) and substitute \( x - \pi \), getting the Taylor polynomial for \( \sin(x - \pi) \) centered at \( \pi \). Then use \( \sin(x) = -\sin(x - \pi) \).

11. (5 points.) A sequence \( (c_n)_{n=1}^\infty \) starts
\[ \left( \frac{1}{2}, -\frac{3}{4}, \frac{5}{8}, -\frac{7}{16}, \frac{9}{32}, \cdots \right). \]
Assuming the pattern continues, find a formula for the general term \( c_n \).

**Solution:** \( c_n = \frac{(-1)^{n+1}(2n - 1)}{2^n} \) for strictly positive integers \( n \).

12. (8 points.) Define a sequence \( (a_n)_{n=1}^\infty \) by \( a_n = (-1)^{n+1}\sin\left(\frac{2}{n}\right) \) for strictly positive integers \( n \). Find its limit (possibly \( \infty \) or \( -\infty \)), giving reasons, or explain why the sequence neither converges nor diverges to \( \infty \) or \( -\infty \). Use mathematically and notationally correct steps.

**Solution:** For \( n = 2, 3, 4, \ldots \), we have \( 0 < \frac{2}{n} \leq \frac{\pi}{2} \), so \( \sin\left(\frac{2}{n}\right) \geq 0 \). Therefore
\[ -\sin\left(\frac{2}{n}\right) \leq a_n \leq \sin\left(\frac{2}{n}\right). \]
Now \( \lim_{n \to \infty} \frac{2}{n} = 0 \), and \( x \mapsto \sin(x) \) is continuous at \( x = 0 \), so \( \lim_{n \to \infty} \sin\left(\frac{2}{n}\right) = 0 \). Similarly
\[ \lim_{n \to \infty} \left(-\sin\left(\frac{2}{n}\right)\right) = 0. \]
Therefore \( \lim_{n \to \infty} a_n = 0 \) by the Squeeze Theorem.

13. (8 points.) Determine whether the series \( \sum_{n=1}^\infty \frac{n}{1 + 2n^2} \) converges or diverges. Be sure to show your reasoning in mathematically and notationally correct steps.
**Solution:** Use the Integral Test. We take \( f(x) = \frac{x}{1 + 2x^2} \). We need to check that \( f(x) \geq 0 \) beyond some point; this is clearly true for all \( x > 0 \). We need to check that \( f \) is nonincreasing beyond some point. We check this using \( f' \). We have

\[
f'(x) = \frac{1 + 2x^2 - x(4x)}{(1 + 2x^2)^2} = \frac{1 - 2x^2}{(1 + 2x^2)^2}.
\]

This expression is clearly negative when \( x > 1 \), so \( f \) is decreasing on \((1, \infty)\).

(A correct solution **must** show that you checked the hypotheses for the Integral Test.)

The substitution \( u = 2x^2 \) gives

\[
\int \frac{x}{1 + 2x^2} \, dx = \frac{1}{4} \ln(1 + 2x^2) + C,
\]

Therefore

\[
\int_{1}^{\infty} \frac{x}{1 + 2x^2} \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{x}{1 + 2x^2} \, dx = \lim_{b \to \infty} \left( \frac{1}{4} \ln(1 + 2x^2) \bigg|_{1}^{b} \right) = \lim_{b \to \infty} \left( \frac{1}{4} \ln(1 + 2b^2) - \frac{1}{4} \ln(1 + 2) \right).
\]

Since \( \lim_{b \to \infty}(1 + 2b^2) = \infty \), we also have \( \lim_{b \to \infty} \frac{1}{4} \ln(1 + 2b^2) = \infty \). So the improper integral diverges. Therefore \( \sum_{n=1}^{\infty} \frac{n}{1 + 2n^2} \) diverges.