Solution. We have:
\[ 0 \leq \Box \]
This completes the solution.

2. (16 points.) Prove the Comparison Test for convergence of an infinite series.

That is, suppose that \( (a_n)_{n \in \mathbb{Z}_{>0}} \) and \( (b_n)_{n \in \mathbb{Z}_{>0}} \) are sequences of real numbers, that for all \( n \in \mathbb{Z}_{>0} \) we have \( 0 \leq a_n \leq b_n \), and that \( \sum_{n \in \mathbb{Z}_{>0}} b_n \) converges. Prove that \( \sum_{n \in \mathbb{Z}_{>0}} a_n \) converges.

Be sure to make clear which previous results you are using in your proof.

Solution. For \( n \in \mathbb{Z}_{>0} \) define
\[ s_n = a_1 + a_2 + \cdots + a_n \quad \text{and} \quad t_n = b_1 + b_2 + \cdots + b_n. \]
It is clear that for all \( n \in \mathbb{Z}_{>0} \) we have \( 0 \leq s_n \leq t_n \). By the Boundedness Criterion for convergence of a series with nonnegative terms, \( (t_n)_{n \in \mathbb{Z}_{>0}} \) is bounded. Therefore \( (s_n)_{n \in \mathbb{Z}_{>0}} \) is bounded. Again by the Boundedness Criterion for convergence of a series with nonnegative terms, \( \sum_{n \in \mathbb{Z}_{>0}} a_n \) converges.

3. (12 points.) Determine, with proof, whether or not the series \( \sum_{n=1}^{\infty} \sin(n^2) \sin\left(\frac{1}{n^2}\right) \) is convergent.

Solution. We use the Comparison Test to prove absolute convergence. Since absolutely convergent series are convergent, this will imply that the series is convergent.

Recall that for all \( x \in [0, \infty) \), we have \( \sin(x) \leq x \). (Proof: Set \( f(x) = x \) for \( x \in \mathbb{R} \). Then \( f(0) = \sin(0) = 0 \), and for all \( x \in [0, \infty) \), \( \sin'(x) = \cos(x) \leq 1 = f'(x) \).

Also, \( n \in \mathbb{Z}_{>0} \) we have \( 0 < \frac{1}{n^2} \leq 1 < \pi \), so \( \sin\left(\frac{1}{n^2}\right) \geq 0 \).

It follows that for all \( n \in \mathbb{Z}_{>0} \) we have
\[ \left| \sin(n^2) \sin\left(\frac{1}{n^2}\right) \right| = |\sin(n^2)| \sin\left(\frac{1}{n^2}\right) \leq |\sin(n^2)| \left(\frac{1}{n^2}\right) \leq \frac{1}{n^2}. \]

We know that \( \sum_{n=0}^{\infty} \frac{1}{n^2} \) is convergent. (The exponent 2 is greater than 1.) So the Comparison Test implies that \( \sum_{n=1}^{\infty} \left| \sin(n^2) \sin\left(\frac{1}{n^2}\right) \right| \) is convergent.

\[ \Box \]
4. (16 points.) Prove directly from the definition of the limit of a sequence that

\[
\lim_{n \to \infty} \left( \frac{1}{n} + \frac{1}{n^2} \right) = 0.
\]

No credit will be given for any argument which is not directly from the definition.

Scratchwork (not required for the solution). Let \( \varepsilon > 0 \). We want

\[
\left| \frac{1}{n} + \frac{1}{n^2} - 0 \right| < \varepsilon,
\]

that is,

\[
\frac{1}{n} + \frac{1}{n^2} < \varepsilon.
\]

It is enough to have

\[
\frac{1}{n} < \frac{\varepsilon}{2} \quad \text{and} \quad \frac{1}{n^2} < \frac{\varepsilon}{2}.
\]

To make the first of these hold, it is enough to require that \( n > 2/\varepsilon \). To make the second hold, it is enough to require that \( n^2 > 2/\varepsilon \). If \( n \geq 1 \), then also \( n^2 \geq n \). So we just want to make sure that \( n \geq 1 \) and \( n > 2/\varepsilon \). This tells us how to choose \( N \).

Solution. Let \( \varepsilon > 0 \). Choose \( N \in \mathbb{Z}^+ \) such that \( N > \max \left( 1, \frac{2}{\varepsilon} \right) \). Let \( n \in \mathbb{Z}^+ \) satisfy \( n \geq N \). Then, using \( n \geq 1 \) at the third step and \( N > \frac{\varepsilon}{2} \) at the fifth step,

\[
\begin{align*}
\left| \frac{1}{n} + \frac{1}{n^2} \right| &= \frac{1}{n} + \frac{1}{n^2} \\
&\leq \frac{1}{N} + \frac{1}{N^2} \\
&\leq \frac{1}{N} + \frac{1}{N} \\
&= \frac{2}{N} < \varepsilon.
\end{align*}
\]

This completes the proof.

5. (a) (10 points.) Prove that for all \( x \in (1, \infty) \) we have

\[
x > \sqrt{2x - 1} > 1.
\]

Hints: You can prove the two inequalities separately. The relation \( x^2 - 2x + 1 = (x - 1)^2 \) may be useful.

Solution for \( \sqrt{2x - 1} > 1 \). Let \( x > 1 \). Then \( 2x - 1 > 1 \) so \( \sqrt{2x - 1} > 1 \).

Solution for \( x > \sqrt{2x - 1} \). Let \( x > 1 \). Then \( x^2 - 2x + 1 = (x - 1)^2 > 0 \), so \( x^2 > 2x - 1 \). Since \( x^2 > 0 \) and \( 2x - 1 > 0 \), it follows that \( \sqrt{x^2} > \sqrt{2x - 1} \). Since \( x > 0 \), we have \( \sqrt{x^2} = x \), so \( x > \sqrt{2x - 1} \).

Alternate solution for both parts. Define functions \( f, g, h : \left( \frac{1}{2}, \infty \right) \to \mathbb{R} \) by

\[
f(x) = x, \quad g(x) = \sqrt{2x - 1}, \quad \text{and} \quad h(x) = 1
\]

for \( x \in \left( \frac{1}{2}, \infty \right) \).

Now \( f, g, \) and \( h \) are all differentiable on \( \left( \frac{1}{2}, \infty \right) \). We have \( g'(x) = (2x - 1)^{-1/2} \). For \( x > 1 \), we have \( 2x - 1 > 1 \), so \( 0 < (2x - 1)^{-1/2} < 1 \). Thus \( f'(x) > g'(x) > h'(x) \) for all \( x \in \left( \frac{1}{2}, \infty \right) \). Since \( f(1) = g(1) = h(1) = 1 \), it follows that \( f(x) > g(x) > h(x) \) for all \( x \in (1, \infty) \), which is what was to be proved.
Second alternate solution for both parts. Since \( x - 1 > 0 \), we have
\[
(x - 1)^2 > 0 > -2(x - 1).
\]
Therefore
\[
x^2 - 2x + 1 > 0 > -2x + 2.
\]
Subtract \(-2x + 1\) from each expression, getting
\[
x^2 > 2x - 1 > 1.
\]
Since all three expressions are in \([0, \infty)\) and \( y \mapsto \sqrt{y} \) is strictly increasing on \([0, \infty)\), it follows that
\[
\sqrt{x^2} > \sqrt{2x - 1} > 1.
\]
Therefore
\[
x > \sqrt{2x - 1} > 1.
\]

Comment 1: The second alternate solution was done by a student.
Comment 2: The second alternate solution was found by working backwards: squaring both sides of (4) to get (3), etc. This is how to find proofs of this sort. However, a correctly written proof goes forwards: it starts with statements which are clearly true, here (1), and uses them to show that succeeding statements are true, here (2), then (3), etc. This is opposite to the order you are used to for solving equations.

(b) (10 points.) Define a sequence \((a_n)_{n \in \mathbb{Z}_{>0}}\) recursively by \(a_1 = 2\) and \(a_{n+1} = \sqrt{2a_n - 1}\) for \(n \in \mathbb{Z}_{>0}\). Prove that \((a_n)_{n \in \mathbb{Z}_{>0}}\) is nondecreasing and bounded below.

(Use Part (a) even if you didn’t solve it.)

Solution. We prove by induction on \(n\) that \(a_n > a_{n+1} > 1\) for all \(n \in \mathbb{Z}_{>0}\).

The base case is \(n = 1\). Then \(a_n = 2\) and \(a_{n+1} = \sqrt{3}\), so the inequality is clear.

For the induction step, let \(n \in \mathbb{Z}_{>0}\), and assume that we know \(a_n > a_{n+1} > 1\). Apply Part (a) with \(x = a_{n+1}\); this is justified by the inequality \(a_{n+1} > 1\). We get \(x > \sqrt{2x - 1} > 1\), that is, \(a_n > a_{n+1} > 1\).

(c) (10 points.) Let \((a_n)_{n \in \mathbb{Z}_{>0}}\) be as in Part (b). Prove that \(\lim_{n \to \infty} a_n\) exists, and find, with proof, its value.

Solution. By Part (b), the sequence \((a_n)_{n \in \mathbb{Z}_{>0}}\) is nondecreasing and bounded below. Therefore \(\lim_{n \to \infty} a_n\) exists and is in \([1, \infty)\). Define \(a = \lim_{n \to \infty} a_n\).

Define \(g: [1, \infty) \to \mathbb{R}\) by \(g(x) = \sqrt{2x - 1}\) for \(x \in [1, \infty)\). Since \(g\) is continuous, we have
\[
\lim_{n \to \infty} g(a_n) = g \left( \lim_{n \to \infty} a_n \right) = g(a).
\]
Also
\[
\lim_{n \to \infty} g(a_n) = \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} a_n = a.
\]
Therefore \(g(a) = a\). That is, \(a = \sqrt{2a - 1}\). Therefore \(a^2 = 2a - 1\), which is \((a - 1)^2 = 0\). This equation has the unique solution \(a = 1\). So \(\lim_{n \to \infty} a_n = 1\).
A correct solution must prove existence of \( \lim_{n \to \infty} a_n \); one can’t just assume the limit exists.
A correct solution must also explicitly mention say that continuity of the function \( x \mapsto \sqrt{2x-1} \) is used at the appropriate step.

6. (16 points.) Find, with proof, an explicit rational number \( r \) such that \( |r - \sqrt[3]{e}| < 10^{-2} \). You need not derive the Taylor polynomial for \( e^x \), you need not find the best choice of \( r \), and you need not simplify your expression for \( r \). You may take it as known that \( e < 3 \).

We use the

Scratchwork (not required for the solution). We will use the value of the Taylor polynomial of a suitable degree for \( f(x) = e^x \) centered at 0 and evaluated at \( \frac{1}{5} \). Since the Taylor polynomials for \( f \) centered at 0 have rational coefficients, this will give a solution.

We use the Lagrange form of the remainder in Taylor’s Theorem; for \( a, x \in \mathbb{R} \), a function \( f \) which is infinitely often differentiable on an interval containing \( a \) and \( x \), and \( n \in \mathbb{Z}_{\geq 0} \), it says that there is \( t \) between between \( a \) and \( x \) such that

\[
(5) \quad f(x) - P_{n,a,f}(x) = R_{n,a,f}(x) = \frac{f^{(n+1)}(t)}{(n+1)!} (x-a)^{n+1}.
\]

We take \( f(x) = e^x \) for \( x \in \mathbb{R} \), we take \( a = 0 \), and we take \( x \) in (5) to be \( \frac{1}{5} \). For any \( n \in \mathbb{Z}_{\geq 0} \) and \( t \in (0, \frac{1}{5}) \), we have \( f^{(n+1)}(t) = e^t \), so

\[
1 \leq f^{(n+1)}(t) \leq e^{1/5} < e < 3,
\]
whence

\[
\frac{1}{(n+1)!} \left( \frac{1}{5} \right)^{n+1} \leq R_{n,0,f} \left( \frac{1}{5} \right) \leq \frac{3}{(n+1)!} \left( \frac{1}{5} \right)^{n+1}.
\]

Since

\[
P_{n,a,f}(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},
\]

this says in particular that

\[
\left| \sqrt[e]{e} - \sum_{k=0}^{n} \frac{1}{k!} \left( \frac{1}{5} \right)^k \right| < \frac{3}{(n+1)!} \left( \frac{1}{5} \right)^{n+1}.
\]

Take

\[
n = 3 \quad \text{and} \quad r = \sum_{k=0}^{3} \frac{1}{k!} \left( \frac{1}{5} \right)^k = 1 + \frac{1}{5} + \frac{1}{5^2 \cdot 2} + \frac{1}{5^3 \cdot 3!}.
\]

Since

\[
\frac{3}{5^4 \cdot 4!} = \frac{1}{54} \cdot \frac{1}{4} = \frac{1}{5000} < \frac{1}{100},
\]

this choice of \( n \) gives the required bound, such that

\[
(6) \quad \frac{3}{(n+1)!} \left( \frac{1}{5} \right)^{n+1} \leq \frac{1}{100}.
\]

and take \( r = P_{n,0,f}(\sqrt[e]{e}) \). It is fairly easy to see that (6) holds for \( n = 3 \) but not for \( n = 2 \).

The choice \( n = 299 \) in the alternate solution was gotten as follows; clearly \( \left( \frac{1}{5} \right)^{n+1} \leq 1 \) and \( (n+1)! > n+1 \). So it is enough to choose \( n \) such that \( 3/(n+1) \leq \frac{1}{100} \). \( \square \)
Solution. Set
\[ r = \sum_{k=0}^{3} \frac{1}{k!} \left( \frac{1}{5} \right)^k = 1 + \frac{1}{5} + \frac{1}{5^2 \cdot 2} + \frac{1}{5^3 \cdot 3!}, \]
which is clearly a rational number.
Define \( f(x) = e^x \) for \( x \in \mathbb{R} \). Then \( r \) is equal to the Taylor polynomial \( P_{3,0,f} \left( \frac{1}{5} \right) \). Therefore
\[ |r - \sqrt[5]{e}| = \left| R_{3,0,f} \left( \frac{1}{5} \right) \right|. \]

Using the Lagrange form of the remainder in Taylor’s Theorem with \( n = 3 \), we see that there is \( t \in (0, \frac{1}{5}) \) such that
\[ R_{3,0,f} \left( \frac{1}{5} \right) = \frac{f^{(4)}(t)}{4!} \left( \frac{1}{5} \right)^4 = \frac{e^t}{5^4 \cdot 4!}. \]

Since \( 0 < e^t < e < 3 \), we have
\[ 0 < R_{3,0,f} < \frac{3}{5^4 \cdot 4!} = \frac{1}{54 \cdot 2 \cdot 4} = \frac{1}{5000} < \frac{1}{100}, \]
from which it follows that \( |r - \sqrt[5]{e}| < \frac{1}{100}. \)

The proof doesn’t quite work for \( n = 2 \). One gets the bound \( 2 \cdot 10^{-2} \).

Alternate solution. This solution is the same as the first solution except in the last paragraph, where we take for the choice of degree of the Taylor polynomial.
Set
\[ r = \sum_{k=0}^{299} \frac{1}{k!} \left( \frac{1}{5} \right)^k, \]
which is clearly a rational number.
Define \( f(x) = e^x \) for \( x \in \mathbb{R} \). Then \( r \) is equal to the Taylor polynomial \( P_{299,0,f} \left( \frac{1}{5} \right) \). Therefore
\[ |r - \sqrt[5]{e}| = \left| R_{299,0,f} \left( \frac{1}{5} \right) \right|. \]

Using the Lagrange form of the remainder in Taylor’s Theorem with \( n = 299 \), we see that there is \( t \in (0, \frac{1}{5}) \) such that
\[ n R_{3,0,f} \left( \frac{1}{5} \right) = 299 R = \sum_{k=0}^{299} \frac{1}{k!} \frac{f^{(300)}(t)}{300!} \left( \frac{1}{5} \right)^{k - 300} = \frac{e^t}{5^4 \cdot 4!}. \]

Since \( 5^{299} > 1 \) and \( 300! > 3000 \), \( 0 < e^t < e < 3 \), and using \( 5^{300} \geq 1 \) and \( 300! \geq 300 \), we have
\[ \frac{3}{5^{299} \cdot 300!} R_{3,0,f} < \frac{3}{5^{300} \cdot 300!} \leq \frac{3}{5 \cdot 300} = \frac{1}{100^2}. \]

So this choice of \( n \) gives the required bound from which it follows that \( |r - \sqrt[5]{e}| < \frac{1}{100}. \)

This choice of \( n \) is of course ridiculously large. However, the wording of the problem doesn’t rule it out, and it makes the estimate at the end trivial.