1. (15 points.) For \( n \in \mathbb{Z}_{>0} \), define a function \( f_n: [1, 2] \to \mathbb{R} \) by \( f_n(x) = \cos(18 + \sqrt[n]{x}) \) for \( x \in [1, 2] \). Determine, with proof, whether there is a function \( f: [1, 2] \to \mathbb{R} \) such that \( f_n \to f \) uniformly on \([1, 2]\). 

Solution. Let \( f: [1, 2] \to \mathbb{R} \) be the constant function \( f(x) = \cos(19) \) for \( x \in [1, 2] \). We claim that \( f_n \to f \) uniformly on \([1, 2]\).

To prove the claim, let \( \varepsilon > 0 \). Since \( \lim_{n \to \infty} \sqrt[n]{2} = 1 \), we can choose \( N \in \mathbb{Z}_{>0} \) such that for all \( n \in \mathbb{Z}_{>0} \) with \( n > N \), we have \( |\sqrt[n]{2} - 1| < \varepsilon \).

Now let \( n \in \mathbb{Z}_{>0} \) satisfy \( n > N \) and let \( x \in [1, 2] \). Then \( 1 \leq \sqrt[n]{x} \leq \sqrt[n]{2} \) because \( x \mapsto \sqrt[n]{x} \) is nondecreasing on \([0, \infty)\). If \( x \neq 1 \), the Mean Value Theorem implies that there is \( t \in [19, 18 + \sqrt[n]{x}] \) such that
\[
\cos(18 + \sqrt[n]{x}) - \cos(19) = -\sin(t)\left(\sqrt[n]{2} - 1\right).
\]

Therefore
\[
|\cos(18 + \sqrt[n]{x}) - \cos(19)| \leq |\sin(t)| \cdot |\sqrt[n]{x} - 1| \leq |\sqrt[n]{2} - 1| < \varepsilon,
\]
as desired. If \( x = 1 \), then
\[
|\cos(18 + \sqrt[n]{x}) - \cos(19)| = 0 < \varepsilon,
\]
as desired. This completes the proof. \( \square \)

2. (12 points) Determine, with proof, whether or not the series \( \sum_{n=1}^{\infty} \left( \sec\left(\frac{1}{n^{2/3}}\right) - 1 \right) \) is convergent.

Solution. Define \( f(x) = \sec(x) - 1 \) for \( x \in \mathbb{R} \setminus \{ n\pi + \frac{\pi}{2} : n \in \mathbb{Z} \} \). Then calculate
\[
f'(x) = \sin(x)[\cos(x)]^{-2} \quad \text{and} \quad f''(x) = [\cos(x)]^{-1} + 2\sin^2(x)[\cos(x)]^{-3}.
\]
In particular,
\[
f(0) = 0, \quad f'(0) = 0, \quad \text{and} \quad f''(0) = 1.
\]
It follows from Theorem 1 in Chapter 20 of Spivak’s book (the result on polynomial approximation) that
\[
0 = \lim_{x \to 0} \frac{f(x) - x^2/2}{x^2} = \lim_{x \to 0} \frac{f(x)}{x^2} - \frac{1}{2}.
\]
Choose \( \delta > 0 \) such that for all \( x \in \mathbb{R} \) with \( 0 < |x| < \delta \) we have
\[
\left| \frac{f(x)}{x^2} - \frac{1}{2} \right| < \frac{1}{2}.
\]
Thus, \( 0 < |x| < \delta \) implies
\[
0 < \sec(x) - 1 < x^2.
\]
Choose $N \in \mathbb{Z}_{>0}$ such that for all $n \in \mathbb{Z}_{>0}$ with $n > N$ we have $n^{-2/3} < \delta$. For $n \in \mathbb{Z}_{>0}$ with $n > N$ we then have
\[
0 < \sec\left(\frac{1}{n^{2/3}}\right) - 1 < \frac{1}{n^{4/3}}.
\]
Since
\[
\sum_{n=N+1}^{\infty} \frac{1}{n^{4/3}}
\]
converges, the Comparison Test implies that
\[
\sum_{n=N+1}^{\infty} \left( \sec\left(\frac{1}{n^{2/3}}\right) - 1 \right)
\]
converges. Therefore
\[
\sum_{n=1}^{\infty} \left( \sec\left(\frac{1}{n^{2/3}}\right) - 1 \right)
\]
converges. \qed

The limit in (1) can also be calculated directly by using L'Hospital's Rule. The derivative calculations are the same.

3. (12 points) Determine, with proof, whether or not the series \(\sum_{n=1}^{\infty} (-1)^n \left(1 - \cos\left(\frac{1}{\sqrt{n}}\right)\right)\) is convergent.

\textbf{Solution.} We use the Alternating Series Test to prove convergence. For $x, y \in [0, \pi]$ with $x \leq y$, we have $\cos(x) \geq \cos(y)$. For all $n \in \mathbb{Z}_{>0}$ we have $1/\sqrt{n} \in [0, \pi]$. Since \(1/\sqrt{n}\) is a nonincreasing sequence, it follows that \((\cos(1/\sqrt{n}))_{n \in \mathbb{Z}_{>0}}\) is a nondecreasing sequence, so that \((1 - \cos(1/\sqrt{n}))_{n \in \mathbb{Z}_{>0}}\) is a nonincreasing sequence. The terms are nonnegative, and, since \(x \mapsto \cos(x)\) is continuous at 0, it follows that
\[
\lim_{n \to \infty} \left(1 - \cos\left(\frac{1}{\sqrt{n}}\right)\right) = 1 - \cos(0) = 0.
\]
The Alternating Series Test now implies convergence of \(\sum_{n=1}^{\infty} (-1)^n \left(1 - \cos\left(\frac{1}{\sqrt{n}}\right)\right)\). \qed

4. Let \((a_n)_{n \in \mathbb{Z}_{>0}}\) be a sequence of complex numbers, and let \(l \in \mathbb{C}\).

\textbf{a.} (8 points.) State the definition of what it means to have \(\lim_{n \to \infty} a_n = l\).

\textbf{Solution.} For every \(\varepsilon > 0\) there is \(N \in \mathbb{Z}_{>0}\) such that for all \(n \in \mathbb{Z}_{>0}\) with \(n > N\) we have \(|a_n - l| < \varepsilon\). \qed
b. (20 points.) Suppose that in fact \( a_n \in \mathbb{R} \) for all \( n \in \mathbb{Z}_{>0} \), that \((a_n)_{n \in \mathbb{Z}_{>0}}\) is nondecreasing and bounded above, and that \( l = \sup_{n \in \mathbb{Z}_{>0}} a_n \). Using the definition in part (a), prove that \( \lim_{n \to \infty} a_n = l \).

**Solution.** Let \( \varepsilon > 0 \). Then \( l - \varepsilon \) is not an upper bound for \( \{a_m : m \in \mathbb{Z}_{>0}\} \). Therefore there exists \( N \in \mathbb{Z}_{>0} \) such that \( a_N > l - \varepsilon \). Let \( n \in \mathbb{Z}_{>0} \) satisfy \( n > N \). Then \( a_n \geq a_N > l - \varepsilon \) since \((a_n)_{n \in \mathbb{Z}_{>0}}\) is nondecreasing, and

\[
\begin{align*}
    a_n &\leq \sup(\{a_m : m \in \mathbb{Z}_{>0}\}) = l. \\
\end{align*}
\]

Thus \( l - \varepsilon < a_n \leq l \). In particular, \(|a_n - l| < \varepsilon\). \( \square \)

5. (20 points.) Let \((a_n)_{n \in \mathbb{Z}_{>0}}\) be a sequence of complex numbers, and suppose \( \lim_{n \to \infty} a_n = 17 \). Prove that the radius of convergence of the power series \( \sum_{n=0}^{\infty} a_n z^n \) is exactly 1.

**Solution.** We claim that if \( z \in \mathbb{C} \) satisfies \(|z| < 1\), then \( \sum_{n=0}^{\infty} a_n z^n \) converges. We prove the claim. Taking \( \varepsilon = 1 \) in the definition of \( \lim_{n \to \infty} a_n = 17 \), choose \( N \in \mathbb{Z}_{>0} \) such that for all \( n \in \mathbb{Z}_{>0} \) with \( n > N \) we have \(|a_n - 17| < 1\). For such \( n \), we then have \(|a_n| < 18\), so

\[
|a_n z^n| = |a_n||z^n| \leq 18|z|^n.
\]

Since \(|z| < 1\), the series \( \sum_{n=N+1}^{\infty} 18|z|^n \) converges, so \( \sum_{n=N+1}^{\infty} |a_n z^n| \) converges by the Comparison Test. Therefore \( \sum_{n=N+1}^{\infty} a_n z^n \) converges. The claim is proved.

We next claim that if \( z \in \mathbb{C} \) satisfies \(|z| > 1\), then \( \sum_{n=0}^{\infty} a_n z^n \) diverges. We prove the claim. Taking \( \varepsilon = 1 \) in the definition of \( \lim_{n \to \infty} a_n = 17 \), choose \( N \in \mathbb{Z}_{>0} \) such that for all \( n \in \mathbb{Z}_{>0} \) with \( n > N \) we have \(|a_n - 17| < 1\). For such \( n \), we then have \(|a_n| > 16\), so \(|a_n z^n| \geq 16|z|^n\). Since \(|z| > 1\), we have \( \lim_{n \to \infty} 16|z|^n = \infty \). Therefore \( a_n z^n \not\to 0 \). The claim follows.

The two claims together imply that radius of convergence of the power series \( \sum_{n=0}^{\infty} a_n z^n \) is 1.

It doesn’t matter what happens when \(|z| = 1\). (However, the series \( \sum_{n=0}^{\infty} a_n z^n \) is easily seen to diverge when \(|z| = 1\).)

6. (20 points.) Define \( f : \mathbb{R} \to \mathbb{R} \) by \( f(x) = \frac{5x}{1 + 3x^2} \) for \( x \in \mathbb{R} \). Compute \( f^{(99)}(0) \) and \( f^{(100)}(0) \).

**Solution.** We know that for all \( y \in (-1,1) \) we have

\[
\sum_{n=0}^{\infty} y^n = \frac{1}{1 - y}.
\]

Therefore for all \( x \in \mathbb{R} \) with \(|x| < 1/\sqrt{3}\), we have

\[
\frac{1}{1 + 3x^2} = \sum_{n=0}^{\infty} (-3x^2)^n = \sum_{n=0}^{\infty} (-1)^n 3^n x^{2n},
\]
so
\[
\frac{5x}{1 + 3x^2} = \sum_{n=0}^{\infty} (-1)^n (5 \cdot 3^n) x^{2n+1}.
\]

By uniqueness of power series representations, this series is the Taylor series for \(f\) centered at 0.

Since the coefficient of \(x^{100}\) in this series is zero, \(f^{(100)}(0) = 0\). To get the coefficient of \(x^{99}\), take \(n = 49\). This gives \(f^{(99)}(0) = 99! \cdot (-1)^{49} \cdot 5 \cdot 3^{49} = -5 \cdot 99! \cdot 3^{49}\).

One doesn’t need the power series to find \(f^{(100)}(0)\). Since \(f\) is odd, repeated application of the chain rule can be used to show that for every even \(n \in \mathbb{Z}_{\geq 0}\) for which \(f^{(n)}(0)\) exists, one has \(f^{(n)}(0) = 0\). However, this method doesn’t help with \(f^{(99)}(0)\).

7. Set \(f(x) = e^x\) for \(x \in \mathbb{R}\).

a. (5 points.) For \(n \in \mathbb{Z}_{\geq 0}\), write down a formula for the Taylor polynomial \(P_{n,0,f}(x)\). You need not justify your answer.

Solution. We have \(P_{n,0,f}(x) = \sum_{k=0}^{n} \frac{x^k}{k!}\).

This follows from the fact that \(f^{(k)}(0) = 1\) for all \(n \in \mathbb{Z}_{\geq 0}\).

b. (18 points.) For \(n \in \mathbb{Z}_{\geq 0}\), let \(R_{n,0,f}(x) = f(x) - P_{n,0,f}(x)\) be the corresponding remainder. Use Taylor’s Theorem to prove that for all \(x \in (0, \infty)\) we have

\[
0 < R_{n,0,f}(x) < \frac{e^x x^{n+1}}{(n+1)!}.
\]

Solution. By Taylor’s Theorem, using the Lagrange form of the remainder, there is \(t \in (0, x)\) such that

\[
R_{n,0,f}(x) = \frac{f^{(n+1)}(t) x^{n+1}}{(n+1)!}.
\]

Now \(f^{(n+1)}(t) = e^t\), and this function is strictly increasing on \((0, x)\), so \(1 < f^{(n+1)}(t) < e^x\). Therefore

\[
0 < \frac{1 \cdot x^{n+1}}{(n+1)!} < R_{n,0,f}(x) < \frac{e^x x^{n+1}}{(n+1)!},
\]

as desired.

c. (7 points.) Use Part (b) to show that \(\frac{8}{3} < e < \frac{67}{24}\). You may take as already known the inequality \(e < 3\).
Solution. In Part (c), add \( P_{n,0,f}(x) \) to both sides, getting
\[
P_{n,0,f}(x) < f(x) < P_{n,0,f}(x) + \frac{e^x x^{n+1}}{(n+1)!}.
\]
Take \( n = 3 \) and \( x = 1 \) in this inequality, and use \( e < 3 \) and \( P_{3,0,f}(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} = \frac{8}{3} \), to get
\[
\frac{8}{3} < e < \frac{8}{3} + \frac{e}{4!} < \frac{8}{3} + \frac{3}{4!} = \frac{67}{24},
\]
as desired. \( \Box \)

8. (10 points) Determine, with proof, whether or not the series \( \sum_{n=2}^{\infty} \frac{1}{n \log(n)} \) is convergent.

Solution. We use the Integral Test. We have
\[
\int_{2}^{x} \frac{1}{t \log(t)} \, dt = \log(\log(t)) \bigg|_{2}^{x} = \log(\log(x)) - \log(\log(2)).
\]
Now \( \lim_{x \to \infty} \log(\log(x)) = \infty \), so \( \int_{2}^{\infty} \frac{1}{t \log(t)} \, dt \) does not converge. The function \( t \mapsto t \log(t) \) is strictly positive and nondecreasing on \([2, \infty)\), and has limit \( \infty \) at \( \infty \). Therefore \( t \mapsto \frac{1}{t \log(t)} \) is nonnegative and nonincreasing on \([2, \infty)\), and has limit 0 at \( \infty \). So the Integral Test shows that \( \sum_{n=2}^{\infty} \frac{1}{n \log(n)} \) does not converge. \( \Box \)

9. (10 points) Determine, with proof, whether or not the series \( \sum_{n=2}^{\infty} \frac{1}{n [\log(n)]^2} \) is convergent.

Solution. We use the Integral Test. We have
\[
\int_{2}^{\infty} \frac{1}{t [\log(t)]^2} \, dt = \lim_{x \to \infty} \int_{2}^{x} \frac{1}{t [\log(t)]^2} \, dt = \lim_{x \to \infty} \left( -\frac{1}{\log(t)} \right)_{2}^{x} = \lim_{x \to \infty} \left( -\frac{1}{\log(x)} + \frac{1}{\log(2)} \right) = \frac{1}{\log(2)}.
\]
So \( \int_{2}^{\infty} \frac{1}{t [\log(t)]^2} \, dt \) converges. The function \( t \mapsto t [\log(t)]^2 \) is strictly positive and nondecreasing on \([2, \infty)\), and has limit \( \infty \) at \( \infty \). Therefore \( t \mapsto \frac{1}{t [\log(t)]^2} \) is nonnegative and nonincreasing on \([2, \infty)\), and has limit 0 at \( \infty \). So the Integral Test shows that \( \sum_{n=2}^{\infty} \frac{1}{n [\log(n)]^2} \) converges. \( \Box \)
10. (8 points.) Evaluate \( \sum_{n=0}^{\infty} \frac{(i\pi)^n}{2^{n-1} \cdot n!} \). Simplify your answer and express it in the form \( a + bi \) with \( a, b \in \mathbb{R} \).

**Solution.** We have
\[
\sum_{n=0}^{\infty} \frac{(i\pi)^n}{2^{n-1} \cdot n!} = 2 \sum_{n=0}^{\infty} \frac{(i\pi/2)^n}{n!} = 2 \exp(i\pi/2) = 2(\cos(\pi/2) + i\sin(\pi/2)) = 2i.
\]
This completes the solution. \( \square \)

11. (8 points.) Evaluate \( \sum_{n=0}^{\infty} \frac{(i\pi)^n}{(n+1)!} \). Simplify your answer and express it in the form \( a + bi \) with \( a, b \in \mathbb{R} \).

**Solution.** We have
\[
\sum_{n=0}^{\infty} \frac{(i\pi)^n}{(n+1)!} = \frac{1}{\pi i} \sum_{n=0}^{\infty} \frac{(i\pi)^{n+1}}{(n+1)!} = \frac{1}{\pi i} \sum_{n=1}^{\infty} \frac{(i\pi)^n}{n!} = \frac{1}{\pi i} (e^{i\pi} - 1) = \frac{1}{\pi i} \left( \cos(\pi) + i\sin(\pi) - 1 \right) = \frac{-2}{\pi i} = \left( \frac{2}{\pi} \right) i.
\]
This completes the solution. \( \square \)

12. Let \( F \) be a ordered field.

a. (10 points.) Let \( x \in F \) be negative. Prove that \( x^2 \) is positive.

**Solution.** We first claim that for any \( y \in F \) we have \( 0 \cdot y = 0 \). To see this, use the distributive law and \( 0 + 0 = 0 \) to get
\[
0 \cdot y + 0 \cdot y = (0 + 0) \cdot y = 0 \cdot y,
\]
and add \(-y \cdot y\) to both sides.

We next claim that any \( y \in F \) we have
\[
(2) \quad (-1) \cdot y = -y.
\]
To see this, use the distributive law and \((-1) + 1 = 0\) to get
\[
(-1) \cdot y + y = (-1) \cdot y + 1 \cdot y = \left[ (-1) + 1 \right] \cdot y = 0 \cdot y = 0.
\]
Now add \(-y\) to both sides.

To prove the statement in the problem, use (2) and commutativity and associativity of multiplication to get
\[
x \cdot x = x \cdot ((-x)) = x \cdot [(-1) \cdot (-x)] = [(-1) \cdot x] \cdot (-x) = (-x) \cdot (-x).
\]
Now \((-x) \cdot (-x)\) is positive because it is the product of two positive elements of \( F \). \( \square \)
b. (10 points.) Prove that the multiplicative identity $1 \in F$ is positive.

*Solution.* One of the field axioms says $1 \neq 0$. So we need to show that $1$ isn’t negative. Suppose, then, that $1$ is negative. We have $1 \cdot 1 = 1$ by the definition of a multiplicative identity. So Part (a) says that $1 = 1 \cdot 1$ is positive. This is a contradiction. $\square$

c. (10 points.) Prove that $F$ has at least one element which is not a square.

*Solution.* We claim that no squares are negative. First, one of the axioms says the product of two positive numbers is positive. So the square of a positive element of $F$ is positive, so not negative. By Part (a), the square of a negative element of $F$ is positive, so not negative. And it follows from the proof of Part (a) that $0^2 = 0$, which is also not negative. The claim is proved.

Part (b) implies that $-1$ is negative. Therefore $-1$ is not a square. $\square$

13. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x) > 0$ for all $x \in (0, \infty)$, and assume that both $f'(0)$ and $f''(0)$ exist. For $n \in \mathbb{Z}_{>0}$, define $a_n = f(1/n)$.

a. (15 points.) If $\sum_{n=1}^{\infty} a_n$ converges, prove that $f(0) = 0$.

*Solution.* The hypotheses imply that $\lim_{n \to \infty} a_n = f(0)$. So if $f(0) \neq 0$, then $\sum_{n=1}^{\infty} a_n$ does not converge because its terms don’t go to zero. $\square$

b. (20 points.) If $\sum_{n=1}^{\infty} a_n$ converges, prove that $f'(0) = 0$.

*Solution.* By Part (a), we must have $f(0) = 0$. Therefore

$$\lim_{h \to 0} \frac{f(h)}{h} = f'(0).$$

We prove that if $f'(0) \neq 0$ then $\sum_{n=1}^{\infty} a_n$ does not converge.

First suppose $f'(0) > 0$. Choose $\delta > 0$ such that for all $h \in \mathbb{R}$ with $0 < |h| < \delta$ we have

$$\left| \frac{f(h)}{h} - f'(0) \right| < \frac{f'(0)}{2}.$$ 

Thus, $0 < |h| < \delta$ implies

$$\frac{f'(0)h}{2} < f(h) < \frac{3f'(0)h}{2}.$$ 

Choose $N \in \mathbb{Z}_{>0}$ such that $N > \frac{1}{\delta}$ for all $n \in \mathbb{Z}_{>0}$ with $n > N$ we then have $\frac{1}{n} < \delta$, so $a_n > f'(0)/(2n)$. Since $\sum_{n=N+1}^{\infty} \frac{1}{2n}$ diverges, the Comparison Test implies that $\sum_{n=N+1}^{\infty} a_n$ diverges. Therefore $\sum_{n=1}^{\infty} a_n$ diverges.

Now suppose $f'(0) < 0$. Apply the case already done to $-f$. We find that $\sum_{n=1}^{\infty} (-a_n)$ diverges. Therefore $\sum_{n=1}^{\infty} a_n$ diverges. $\square$
c. (25 points.) Assume that \( f(0) = f'(0) = 0 \), and prove that \( \sum_{n=1}^{\infty} a_n \) converges.

Hint: By Taylor’s Theorem, there is a function \( R: \mathbb{R} \to \mathbb{R} \) such that \( \lim_{x \to 0} x^{-2} R(x) = 0 \) and \( f(x) = \frac{1}{2} f''(0) x^2 + R(x) \) for all \( x \in \mathbb{R} \).

**Solution.** Let \( R \) be as in the hint. Apply the definition of the limit with \( \varepsilon = 1 \), getting \( \delta > 0 \) such that whenever \( x \in \mathbb{R} \) satisfies \( 0 < |x| < \delta \), then \( |x^2 R(x)| < 1 \), equivalently, \( |R(x)| < x^2 \). Choose \( N \in \mathbb{Z}^+ > 0 \) such that \( \frac{1}{N} < \delta \). For \( n \in \mathbb{Z}^+ > N \) we then have

\[
|a_n| = \left| \left( \frac{1}{2} \right) f''(0) \left( \frac{1}{n^2} \right) + R \left( \frac{1}{n} \right) \right| \leq \left( \frac{1}{2} \right) |f''(0)| \left( \frac{1}{n^2} \right) + \frac{1}{n^2} = \left[ \left( \frac{1}{2} \right) |f''(0)| + 1 \right] \frac{1}{n^2}.
\]

Since \( \sum_{n=N+1}^{\infty} \frac{1}{n^2} \) converges, so does

\[
\sum_{n=N+1}^{\infty} \left[ \left( \frac{1}{2} \right) |f''(0)| + 1 \right] \frac{1}{n^2}.
\]

The Comparison Test now implies that \( \sum_{n=N+1}^{\infty} |a_n| \) converges. Therefore \( \sum_{n=1}^{\infty} |a_n| \) converges, so that \( \sum_{n=1}^{\infty} a_n \) converges. \( \square \)

14. (12 points.) Find, with proof, the radius of convergence of the power series

\[
\sum_{n=0}^{\infty} \exp(n + i(n^2 + n + 1)) z^n.
\]

**Solution.** The series clearly converges when \( z = 0 \). For \( z \neq 0 \), we use the Ratio Test. Recall that if \( \alpha \) is real then \( |e^{i\alpha}| = 1 \). Using this fact at the second step, we have

\[
\lim_{n \to \infty} \frac{\left| \exp(n + 1 + i[(n + 1)^2 + (n + 1) + 1]) z^{n+1} \right|}{\left| \exp(n + i(n^2 + n + 1)) z^n \right|} = \lim_{n \to \infty} \frac{|e^{n+1}| \cdot |\exp(i[(n + 1)^2 + (n + 1) + 1])| \cdot |z|^{n+1}}{|e^n| \cdot |\exp(i(n^2 + n + 1))| \cdot |z|^n} = \lim_{n \to \infty} e|z|.
\]

If \( |z| < \frac{1}{e} \) then \( e|z| < 1 \), so the series converges by the Ratio Test. If \( |z| > \frac{1}{e} \) then \( e|z| > 1 \), so the series diverges by the Ratio Test. Therefore the radius of convergence is \( \frac{1}{e} \). \( \square \)

It does not matter what happens when \( |z| = \frac{1}{e} \).

15. (20 points.) Let \( (a_n)_{n \in \mathbb{Z}^+} \) be a sequence of complex numbers. Prove directly from the definition of the limit of a sequence that if \( \lim_{n \to \infty} a_n \) exists then \( (a_n)_{n \in \mathbb{Z}^+} \) is bounded.
Solution. Set \( l = \lim_{n \to \infty} a_n \). Apply the definition of the limit of a sequence with \( \varepsilon = 1 \) to get \( N \in \mathbb{Z}_{>0} \) such that for all \( n \in \mathbb{Z}_{>0} \) with \( n > N \) we have \(|a_n - l| < 1\). Define

\[ M = \max(|l| + 1, |a_1|, |a_2|, \ldots, |a_N|). \]

We claim that for all \( n \in \mathbb{Z}_{>0} \) we have \(|a_n| \leq M\).

To prove the claim, let \( n \in \mathbb{Z}_{>0} \). If \( n \leq N \) then \(|a_n| \leq M\) by the definition of \( M \). If \( n > N \), then \(|a_n - l| < 1\), so \(|a_n| < |l| + 1 \leq M\). The claim is proved, and the solution is complete. \( \square \)

16. (15 points.) For \( n \in \mathbb{Z}_{>0} \), define a function \( f_n : [0, \infty) \to \mathbb{R} \) by \( f_n(x) = \arctan(nx^2) \) for \( x \in [0, \infty) \). Determine, with proof, whether there is a function \( f : [0, \infty) \to \mathbb{R} \) such that \( f_n \to f \) uniformly on \([0, \infty)\).

Solution. We claim that there is no such function \( f \).

Define \( f : [0, \infty) \to \mathbb{R} \) by

\[ f(x) = \begin{cases} 
0 & x = 0 \\
\frac{\pi}{2} & x \in (0, \infty).
\end{cases} \]

We show that \( f_n \to f \) pointwise. If \( x = 0 \) then \( f_n(x) = 0 \) for all \( n \in \mathbb{Z}_{>0} \), so \( \lim_{n \to \infty} f_n(x) = 0 = f(0) \). If \( x \in (0, \infty) \), then \( \lim_{n \to \infty} nx^2 = \infty \). Since \( \lim_{y \to \infty} \arctan(y) = \frac{\pi}{2} \), with \( y \) running through real numbers, it follows that

\[ \lim_{n \to \infty} f_n(x) = \lim_{y \to \infty} \arctan(nx^2) = \lim_{y \to \infty} \arctan(y) = \frac{\pi}{2} = f(x). \]

Thus \( f_n \to f \) pointwise.

If \( (f_n)_{n \in \mathbb{Z}_{>0}} \) converges uniformly to any function, that function must clearly be \( f \). However, \( f \) is not continuous (because \( \lim_{y \to 0^+} f(y) = \frac{\pi}{2} \neq f(0) \)), and the uniform limit of continuous functions is continuous. So \( (f_n)_{n \in \mathbb{Z}_{>0}} \) does not converge uniformly to \( f \), and therefore does not converge uniformly to any function. \( \square \)

It is also possible to prove the failure of uniform convergence directly from the definition.

17. (15 points.) For \( n \in \mathbb{Z}_{>0} \), define a function \( f_n : [0, \infty) \to \mathbb{R} \) by \( f_n(x) = \arctan(2 + nx^2) \) for \( x \in [0, \infty) \). Determine, with proof, whether there is a function \( f : [0, \infty) \to \mathbb{R} \) such that \( f_n \to f \) uniformly on \([0, \infty)\).

Solution. We claim that there is no such function \( f \).

Define \( f : [0, \infty) \to \mathbb{R} \) by

\[ f(x) = \begin{cases} 
\arctan(2) & x = 0 \\
\frac{\pi}{2} & x \in (0, \infty).
\end{cases} \]
We show that \( f_n \to f \) pointwise. If \( x = 0 \) then \( f_n(x) = \arctan(2) \) for all \( n \in \mathbb{Z}_{>0} \), so \( \lim_{n \to \infty} f_n(x) = \arctan(2) = f(0) \). If \( x \in (0, \infty) \), then \( \lim_{n \to \infty} nx^2 = \infty \). Since \( \lim_{y \to \infty} \arctan(y) = \frac{\pi}{2} \), with \( y \) running through real numbers, it follows that

\[
\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \arctan(nx^2) = \lim_{y \to \infty} \arctan(y) = \frac{\pi}{2} = f(x).
\]

Thus \( f_n \to f \) pointwise.

If \( (f_n)_{n \in \mathbb{Z}_{>0}} \) converges uniformly to any function, that function must clearly be \( f \). However, \( f \) is not continuous (because \( \lim_{y \to 0^+} f(y) = \frac{\pi}{2} \neq f(0) \)), and the uniform limit of continuous functions is continuous. So \( (f_n)_{n \in \mathbb{Z}_{>0}} \) does not converge uniformly to \( f \), and therefore does not converge uniformly to any function.

It is also possible to prove the failure of uniform convergence directly from the definition.

18. (15 points.) For \( n \in \mathbb{Z}_{>0} \), define a function \( f_n : [3, \infty) \to \mathbb{R} \) by \( f_n(x) = \arctan(2 + nx^2) \) for \( x \in [3, \infty) \). Determine, with proof, whether there is a function \( f : [3, \infty) \to \mathbb{R} \) such that \( f_n \to f \) uniformly on \([3, \infty) \).

**Solution.** Define \( f(x) = \frac{\pi}{2} \) for \( x \in [3, \infty) \). We claim that \( f_n \to f \) uniformly on \([3, \infty) \).

To prove the claim, let \( \varepsilon > 0 \). Use \( \lim_{y \to \infty} \arctan(y) = \frac{\pi}{2} \) with \( y \) running through real numbers, and \( \lim_{n \to \infty}(9n + 2) = \infty \), to get \( \lim_{n \to \infty} \arctan(9n + 2) = \frac{\pi}{2} \). Choose \( N \in \mathbb{Z}_{>0} \) such that for all \( n \in \mathbb{Z}_{>0} \) with \( n > N \) we have \( |\arctan(9n + 2) - \frac{\pi}{2}| < \varepsilon \). Now let \( x \in [3, \infty) \). Since \( y \mapsto \arctan(y) \) is nondecreasing on \([3, \infty) \) (in fact, on all of \( \mathbb{R} \)), and \( x^2 \geq 9 \), we have

\[
\arctan(9n + 2) \leq \arctan(2 + nx^2) \leq \frac{\pi}{2}.
\]

Therefore

\[
|\arctan(2 + nx^2) - \frac{\pi}{2}| \leq |\arctan(9n + 2) - \frac{\pi}{2}| < \varepsilon.
\]

This completes the solution.

The next solution differs only in the arrangement of the choice of \( N \).

**Alternate solution.** Define \( f(x) = \frac{\pi}{2} \) for \( x \in [3, \infty) \). We claim that \( f_n \to f \) uniformly on \([3, \infty) \).

To prove the claim, let \( \varepsilon > 0 \). Use \( \lim_{y \to \infty} \arctan(y) = \frac{\pi}{2} \) to choose \( M \in (0, \infty) \) such that for all \( y \in (M, \infty) \) we have \( |\arctan(y) - \frac{\pi}{2}| < \varepsilon \). Choose \( N \in \mathbb{Z}_{>0} \) such that \( N > \frac{M}{9} \). For \( n \in \mathbb{Z}_{>0} \) with \( n > N \) we then have \( 9n + 2 > 9n > M \), so \( |\arctan(9n + 2) - \frac{\pi}{2}| < \varepsilon \). Now let \( x \in [3, \infty) \). Since \( y \mapsto \arctan(y) \) is nondecreasing on \([3, \infty) \) (in fact, on all of \( \mathbb{R} \)), we have

\[
\arctan(9n + 2) \leq \arctan(2 + nx^2) \leq \frac{\pi}{2}.
\]

Therefore

\[
|\arctan(2 + nx^2) - \frac{\pi}{2}| \leq |\arctan(9n + 2) - \frac{\pi}{2}| < \varepsilon.
\]

This completes the solution.