1. (1 point) (10 points.) Define \( f : \mathbb{R} \to \mathbb{R} \) by \( f(x) = \sin(2x) \) for \( x \in \mathbb{R} \). Use the definition to find the Taylor polynomial \( P_{5,0,f}(x) \).

2. (16 points.) Prove the Comparison Test for convergence of an infinite series.
   That is, suppose that \( (a_n)_{n \in \mathbb{Z}_>0} \) and \( (b_n)_{n \in \mathbb{Z}_>0} \) are sequences of real numbers, that for all \( n \in \mathbb{Z}_>0 \) we have \( 0 \leq a_n \leq b_n \), and that \( \sum_{n \in \mathbb{Z}_>0} b_n \) converges. Prove that \( \sum_{n \in \mathbb{Z}_>0} a_n \) converges.
   Be sure to make clear which previous results you are using in your proof.

3. (12 points.) Determine, with proof, whether or not the series \( \sum_{n=1}^{\infty} \sin(n^2) \sin\left(\frac{1}{n^2}\right) \) is convergent.

4. (16 points.) Prove directly from the definition of the limit of a sequence that
   \[
   \lim_{n \to \infty} \left( \frac{1}{n} + \frac{1}{n^2} \right) = 0.
   \]
   No credit will be given for any argument which is not directly from the definition.

5. (a) (10 points.) Prove that for all \( x \in (1, \infty) \) we have
   \[
   x > \sqrt{2x - 1} > 1.
   \]
   Hints: You can prove the two inequalities separately. The relation \( x^2 - 2x + 1 = (x - 1)^2 \) may be useful.

   (b) (10 points.) Define a sequence \( (a_n)_{n \in \mathbb{Z}_>0} \) recursively by \( a_1 = 2 \) and \( a_{n+1} = \sqrt{2a_n - 1} \) for \( n \in \mathbb{Z}_>0 \). Prove that \( (a_n)_{n \in \mathbb{Z}_>0} \) is nondecreasing and bounded below.
   (Use Part (a) even if you didn’t solve it.)

   (c) (10 points.) Let \( (a_n)_{n \in \mathbb{Z}_>0} \) be as in Part (b). Prove that \( \lim_{n \to \infty} a_n \) exists, and find, with proof, its value.

6. (16 points.) Find, with proof, an explicit rational number \( r \) such that \( |r - \sqrt[3]{e}| < 10^{-2} \). You need not derive the Taylor polynomial for \( e^x \), you need not find the best choice of \( r \), and you need not simplify your expression for \( r \). You may take it as known that \( e < 3 \).