Basic instructions: Unless otherwise specified, all claims must be proved, including properties claimed for counterexamples, unless otherwise specified, just as in homework. Theorems in the book may be used, but (unless otherwise specified) not theorems from homework.

Book, notes, calculators, and all other electronic devices are prohibited, except as described below. The only allowed materials, except as described below, are pens or pencils. Space will be provided on the exam to write answers, and scratch paper will be provided.

Do not write anything less than 1/2 inch from any edge of any page which is turned in.

Allowed notes: for each of the theorems on the list of proofs to know for the exam, you may have a “crib sheet” of at most 144 characters. (That is, 144 characters per proof, but they are not transferrable between proofs.)

Unless otherwise specified, I will follow the usual conventions on the symbols used in limits at $\infty$. That is, I will assume that $\lim_{n \to \infty} f(n) = L$ refers to the limit as $n$ runs through positive integers, and that $\lim_{x \to \infty} f(x) = L$ refers to the limit as $x$ runs through real numbers. If, in your solution to some problem, you don’t specify, I will make some assumption depending on the notation and context, and there will be no appeal except in cases in which I actually misread something.

Problems.

1. (10 points.) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f^{(n)}(0)$ exists for $n = 0, 1, 2, \ldots$. Fix $c \in \mathbb{R}$ and $k \in \{1, 2, \ldots\}$. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = f(cx^k)$ for all $x \in \mathbb{R}$. Prove that for $n = 0, 1, 2, \ldots$, the Taylor polynomial $P_{kn,0,g}(x)$ is given by $P_{n,0,f}(cx^k)$ for $x \in \mathbb{R}$.

2. (8 points/part.) For each of the following functions $f$, calculate the Taylor polynomial $P_{100,0,f}(x)$ (degree 100, center at 0, for the function $f$). You may use the result of Problem 1.
   a. $f(x) = \frac{1}{1-x}$ for $x \in (-1, 1)$.
   b. $f(x) = \frac{1}{1+x^2}$ for $x \in \mathbb{R}$.
   c. $f(x) = \arctan(x)$ for $x \in \mathbb{R}$.
   d. $f(x) = \arctan(3x^2)$ for $x \in \mathbb{R}$.

3. (8 points.) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \arctan(3x^2)$ for $x \in \mathbb{R}$. Calculate $f^{(98)}(0)$. You may use the result of Problems 1 and 2d.

4. (10 points/part.) Let $f(x) = \log(1 + x)$ for $x \in (-1, \infty)$. 
a. Use Taylor’s Theorem to prove that, if $x > 0$, then the remainder $R_{2,0,f}(x)$ satisfies $0 < R_{2,0,f}(x) < \frac{x^3}{3}$ for all $x > 0$.

b. Give, with proof, an estimate for $\log(1 + \frac{1}{10})$ which is accurate to within $\frac{1}{3000}$. (Express your answer as a rational number; simplification is not required.)

5. (10 points/part.) Let $(a_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence in $\mathbb{R}$, and let $l \in \mathbb{R}$.

a. Prove that if $(a_n)_{n \in \mathbb{Z}_{>0}}$ converges to $l$, then every subsequence of $(a_n)_{n \in \mathbb{Z}_{>0}}$ also converges to $l$.

b. Prove by example that the converse to part (a) is false.

6. (10 points.) Prove that $\lim_{n \to \infty} n^{1/n} = 1$. (This need not be done directly from the definition.)