MATH 281 (PHILLIPS): SOLUTIONS TO THE FINAL EXAM

Problem 1 (1 point)**.** You have stolen a time machine. You are going back in time to assassinate which of these people:

(a) The inventor of cross products.
(b) The inventor of the osculating circle to a curve.
(c) The inventor of Lagrange multipliers.
(d) The inventor of `WeBWorK`.

Solution. I hope you don't choose any of the mathematicians! □

Problem 2 (18 points)**.** Find an equation for the tangent plane to the surface

$$xy + \sin(xz) + 3z = y^3 + 5$$

at the point $(0, 1, 2)$.

Solution. Set $f(x, y, z) = xy + \sin(xz) + 3z - y^3$, so that the surface in question is the level surface $f(x, y, z) = 5$. For the normal vector to the plane, we take $\mathbf{n} = \mathbf{grad}(f)(0, 1, 2)$. So begin by calculating

$$\mathbf{grad}(f)(x, y, z) = \langle f_x(x, y, z),\ f_y(x, y, z),\ f_z(x, y, z)\rangle = \big\langle y + z\cos(xz),\ x - 3y^2,\ x\cos(xz) + 3\big\rangle$$

and

$$\mathbf{grad}(f)(0, 1, 2) = \langle 3,\ -3,\ 3\rangle.$$

The plane goes through the point $(0, 1, 2)$, so we get the equation

$$\langle 3,\ -3,\ 3\rangle \cdot (\langle x, y, z\rangle - \langle 0, 1, 2\rangle) = 0,$$

that is, $3x - 3(y - 1) + 3(z - 2) = 0$. We simplify this to $3x - 3y + 3z = 3$, and then to $x - y + z = 1$. □

Problem 3 (12 points)**.** Let k be a function such that $k'(t) = \cos(\sin(t))$. Suppose

$$f(x, y, z) = e^{2x} + \arctan(z)k(xy^2 - x^2y) - \ln(15).$$

Find $D_2 f(x, y, z)$.

Solution. We have

$$\begin{aligned} D_2 f(x, y, z) &= \frac{\partial}{\partial y}\big(e^{2x}\big) + \frac{\partial}{\partial y}\big(\arctan(z)k(xy^2 - x^2y)\big) - \frac{\partial}{\partial y}\big(\ln(15)\big) \\ &= 0 + \arctan(z)k'(xy^2 - x^2y)\frac{\partial}{\partial y}\big(xy^2 - x^2y\big) - 0 \\ &= \arctan(z)\cos\big(\sin(xy^2 - x^2y)\big)(2xy - x^2). \end{aligned}$$

□

Date: 7 December 2020.

Problem 4 (20 points)**.** Find and classify (as local maximums, local minimums, or saddle points) all critical points of the function $f(x, y) = 3x - x^3 - 3xy^2$.

Solution. We calculate:

$$f_x(x, y) = 3 - 3x^2 - 3y^2 \qquad \text{and} \qquad f_y(x, y) = -6xy.$$

These exist everywhere. To find the critical points, we therefore need to solve the simultaneous equations

$$3 - 3x^2 - 3y^2 = 0 \qquad \text{and} \qquad -6xy = 0,$$

equivalently

$$x^2 + y^2 = 1 \qquad \text{and} \qquad xy = 0.$$

The second equation gives $x = 0$ or $y = 0$. If $x = 0$ then the first equation gives $y = \pm 1$, and if $y = 0$ then the first equation gives $x = \pm 1$. Thus, $(0, 1)$, $(0, -1)$, $(1, 0)$, and $(-1, 0)$ are all possible critical points. One checks that both partial derivatives really do vanish at each of these points.

We now apply the second derivative test. We have

$$f_{xx}(x, y) = -6x, \qquad f_{yy}(x, y) = -6x, \qquad \text{and} \qquad f_{xy}(x, y) = -6y.$$

So $D(0, 1) = (0)(0) - (-6)^2 = -36 < 0$, and f has a saddle point at $(0, 1)$. Also, $D(0, -1) = (0)(0) - (6)^2 = -36 < 0$, and f has a saddle point at $(0, -1)$. Next, $D(1, 0) = (-6)(-6) - (0)^2 = 36 > 0$ and $f_{xx}(1, 0) = -6 < 0$, so f has a local maximum at $(1, 0)$. Finally, $D(-1, 0) = (6)(6) - (0)^2 = 36 > 0$ and $f_{xx}(-1, 0) = 6 > 0$, so f has a local minimum at $(-1, 0)$. □

Problem 5 (10 points)**.** Let f, x, and y be differentiable functions of two variables, and let $H(s, t) = f(x(s, t), y(s, t))$. Suppose that

$$\begin{gathered} x(5, 2) = 2, \qquad D_1x(5, 2) = -3, \qquad D_2x(5, 2) = 5, \\ y(5, 2) = 5, \qquad D_1y(5, 2) = -4, \qquad D_2y(5, 2) = 2, \\ f(5, 2) = -11, \qquad D_1f(5, 2) = 4, \qquad D_2f(5, 2) = 3, \\ f(2, 5) = -1, \qquad D_1f(2, 5) = -2, \qquad \text{and} \qquad D_2f(2, 5) = 7. \end{gathered}$$

Find $D_2H(5, 2)$.

Solution. We have

$$\begin{aligned} D_2H(5, 2) &= D_1f(x(5, 2), y(5, 2))D_2x(5, 2) + D_2f(x(5, 2), y(5, 2))D_2y(5, 2) \\ &= D_1f(2, 5)D_2x(5, 2) + D_2f(2, 5)D_2y(5, 2) = (-2)(5) + (7)(2) = 4. \end{aligned}$$

□

Problem 6 (15 points)**.** Find an equation for the normal plane to the parametric curve given by $\mathbf{r}(t) = \left\langle 4\sin(2t - 4) + 3,\ t,\ t^2 - 5t \right\rangle$ at the point $(3, 2, -6)$.

Solution. We have $\mathbf{r}(t) = (3, 2, -6)$ when $t = 2$. Start by calculating:

$$\mathbf{r}'(t) = \left\langle 8\cos(2t - 4),\ 1,\ 2t - 5 \right\rangle.$$

So $\mathbf{r}'(2) = \langle 8, 1, -1 \rangle$. Take this to be the normal vector of the plane. Then an equation for the plane is

$$\langle 8, 1, -1 \rangle \cdot (\langle x, y, z \rangle - \langle 3, 2, -4 \rangle) = 0.$$

This is $8(x-3)+(y-2)-(z+6)=0$, which simplifies to $8x+y-z-32=0$. (The simplification is *required.*) ☐

Problem 7 (20 points)**.** Find the maximum and minimum values of $x-3y$ on the circle $x^2+y^2=10$.

Solution. We set

$$f(x,y)=x-3y \qquad \text{and} \qquad g(x,y)=x^2+y^2.$$

The Lagrange multiplier equations are then

$$g(x,y)=10, \qquad D_1f(x,y)=\lambda D_1g(x,y), \qquad \text{and} \qquad D_2f(x,y)=\lambda D_2g(x,y).$$

Putting in what the functions are, we get

$$x^2+y^2=10, \qquad 1=\lambda(2x), \qquad \text{and} \qquad -3=\lambda(2y).$$

The last two equations imply $\lambda\neq 0$, and give

$$x=\frac{1}{2\lambda} \qquad \text{and} \qquad y=-\frac{3}{2\lambda}. \tag{1}$$

Therefore

$$10=\left(\frac{1}{2\lambda}\right)^2+\left(-\frac{3}{2\lambda}\right)^2=\frac{1^2+(-3)^2}{(2\lambda)^2}=\frac{10}{(2\lambda)^2}.$$

So $(2\lambda)^2=1$, whence $\lambda=\frac{1}{2}$ or $\lambda=-\frac{1}{2}$.

If $\lambda=\frac{1}{2}$, then $x=1$ and $y=-3$. If $\lambda=-\frac{1}{2}$, then $x=-1$ and $y=3$. Thus, we must compare the values

$$f(1,\,-3)=1-3(-3)=10 \qquad \text{and} \qquad f(-1,\,3)=-1-3(3)=-10.$$

So the maximum value is 10, at $(1,\,-3)$, and the minimum value is -10, at $(-1,\,3)$. ☐

Alternate solution. Proceed as in the first solution through (1). Combine the equations there to get $y=-3x$. Substitute this in $x^2+y^2=10$ to get

$$10=x^2+y^2=x^2+(-3x)^2=10x^2,$$

so that $x=1$ or $x=-1$. If $x=1$ then $y=-3x$ implies $y=-3$, and if $x=-1$ then $y=-3x$ implies $y=3$. Finish as in the first solution. ☐

Problem 8 (14 points)**.** At position (x,y,z), measured in meters, and if $-5\leq x\leq 5$, $-5\leq y\leq 5$, and $0\leq z\leq 10$, the relative humidity of the air in percentage points is given by

$$H(x,y,z)=58+\frac{1}{2000}\big(x^3+2xy^2-3z\big).$$

A mosquito, looking for an animal to bite, is located at $(-2,3,5)$. If it flies in a straight line towards the point $(0,-3,2)$ at one meter per minute, at what rate does the relative humidity of the air it is in change? Be sure to include units.

Solution. We need the directional derivative in the direction of the unit vector $\mathbf{u}$ in the direction of

$$\mathbf{v}=\langle 0,\,-3,\,2\rangle-\langle -2,\,3,\,5\rangle=\langle 2,\,-6,\,-3\rangle.$$

We have

$$\|\mathbf{v}\|=\sqrt{2^2+(-6)^2+(-3)^2}=\sqrt{49}=7.$$

So
$$\mathbf{u} = \left(\frac{1}{\|\mathbf{v}\|}\right)\mathbf{v} = \left\langle \frac{2}{7},\, -\frac{6}{7},\, -\frac{3}{7} \right\rangle.$$

To find the directional derivative, we find the gradient:
$$\mathbf{grad}(H)(x,y,z) = \frac{1}{2000}\big\langle 3x^2 + 2y^2,\, 4xy,\, -3 \big\rangle,$$
so
$$\mathbf{grad}(H)(-2,3,5) = \frac{1}{2000}\langle 30,\, -24,\, -3\rangle.$$
Then the rate of change is
$$\begin{aligned} D_{\mathbf{u}}H(-2,3,5) &= \mathbf{grad}(H)(-2,3,5)\cdot\mathbf{u} \\ &= \frac{1}{2000}\left((30)\left(\frac{2}{7}\right) + (-24)\left(-\frac{6}{7}\right) + (-3)\left(-\frac{3}{7}\right)\right) = \frac{213}{14000}, \end{aligned}$$
in percentage points of relative humidity per minute. (This is approximately 0.01521423 percentage points of relative humidity per minute.) □

Problem 9 (12 points)**.** A parametrized curve is give by $\mathbf{r}(t) = \langle t,\, 15,\, t^3\rangle$. Find its curvature at time t.

Solution. It is easiest to use the formula
$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$
We compute
$$\mathbf{r}'(t) = \langle 1,\, 0,\, 3t^2\rangle \qquad \text{and} \qquad \mathbf{r}''(t) = \langle 0,\, 0,\, 6t\rangle.$$
Therefore
$$\|\mathbf{r}'(t)\| = \sqrt{1+9t^4},$$
$$\mathbf{r}'(t)\times\mathbf{r}''(t) = \big\langle (0)(6t) - (3t^2)(0),\, (6t)(0) - (1)(6t),\, (1)(0) - (0)(0)\big\rangle = \langle 0,\, -6t,\, 0\rangle,$$
and
$$\|\mathbf{r}'(t)\times\mathbf{r}''(t)\| = 6|t|.$$
So
$$\kappa(t) = \frac{6|t|}{(1+9t^4)^{3/2}}.$$
(It is wrong with t in place of $|t|$ in the numerator.) □

Alternate solution (not recommended). We use the formula
$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}.$$
We compute
$$\mathbf{r}'(t) = \langle 1,\, 0,\, 3t^2\rangle \qquad \text{and} \qquad \|\mathbf{r}'(t)\| = \sqrt{1+9t^4}.$$
So
$$\mathbf{T}(t) = \frac{1}{\|\mathbf{r}'(t)\|}\mathbf{r}'(t) = \big\langle (1+9t^4)^{-1/2},\, 0,\, 3t^2(1+9t^4)^{-1/2}\big\rangle. \tag{2}$$

Thus

$$\begin{aligned}\mathbf{T}'(t) &= \left\langle -\frac{1}{2}(36t^3)(1+9t^4)^{-3/2},\ 0,\ 6t(1+9t^4)^{-1/2} + 3t^2\left(-\frac{1}{2}\right)(36t^3)(1+9t^4)^{-3/2} \right\rangle \\ &= \left\langle -18t^3(1+9t^4)^{-3/2},\ 0,\ 6t(1+9t^4)(1+9t^4)^{-3/2} - 54t^5(1+9t^4)^{-3/2} \right\rangle \\ &= \frac{6t}{(1+9t^4)^{3/2}}\langle -3t^2,\ 0,\ 1\rangle.\end{aligned}$$

So

$$\|\mathbf{T}'(t)\| = \frac{6|t|}{(1+9t^4)^{3/2}}\sqrt{(-3t)^2+1^2} = \frac{6|t|}{1+9t^4}.$$

Therefore

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \left(\frac{6|t|}{1+9t^4}\right)\left(\frac{1}{\sqrt{1+9t^4}}\right) = \frac{6|t|}{(1+9t^4)^{3/2}}.$$

(It is wrong with t in place of $|t|$ in the numerator.) □

The form of $\mathbf{T}(t)$ used in (2) is easier to differentiate than

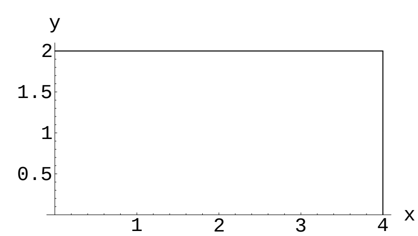
$$\mathbf{T}(t) = \left\langle \frac{1}{\sqrt{1+9t^4}},\ 0,\ \frac{3t^2}{\sqrt{1+9t^4}} \right\rangle.$$

Problem 10 (25 points)**.** Find the maximum and minimum values of the function

$$f(x,y) = (x-y)^2 + (x-4)^2 - 1$$

on the closed rectangle with vertices $(0,0)$, $(0,2)$, $(4,0)$, and $(4,2)$.

Solution. Call the region Y. Here is a picture:



We start by finding the critical points. We calculate:

$$f_x(x,y) = 2(x-y) + 2(x-4) \qquad \text{and} \qquad f_y(x,y) = -2(x-y).$$

(You are better off *not* multiplying out either the formula for the function or the formulas for these partial derivatives.) These exist everywhere. To find the critical points, we therefore need to solve the simultaneous equations

$$2(x-y) + 2(x-4) = 0 \qquad \text{and} \qquad -2(x-y) = 0.$$

The second equation implies $y = x$, and the first equation then implies $x = 4$. So there is a unique solution, namely $x = 4$ and $y = 4$. The point $(4,4)$ is not in Y, so we ignore it. (If you use this point, you will get the *wrong* answer to the problem, because $f(4,4) = -1$ is less than any value f has on Y.)

The boundary of Y consists of four straight line segments, namely the x-axis for $0 \leq x \leq 4$ (the bottom), the line $y = 2$ for $0 \leq x \leq 4$ (the top), the line y-axis for $0 \leq y \leq 2$ (the left side), and the line $x = 4$ for $0 \leq y \leq 2$ (the right side).

For the bottom part, choose the parametrization $x = t$ and $y = 0$ for $0 \leq t \leq 4$. We then need the maximum and minimum values of

$$h_1(t) = f(t, 0) = t^2 + (t-4)^2 - 1 = 2t^2 - 8t + 15$$

for $0 \leq t \leq 4$. Then $h_1'(t) = 4t - 8$, which is zero when $t = 2$. This number is in the interval $[0, 4]$, and $h_1(2) = 7$. (It is easiest to calculate this using the formula $h_1(t) = t^2 + (t-4)^2 - 1$.) It remains to check the endpoints. At the endpoint $t = 0$ we get $h_1(0) = 15$, and at the endpoint $t = 4$ we get $h_1(4) = 15$. (Again, it is easiest to calculate $h_1(4)$ using the formula $h_1(t) = t^2 + (t-4)^2 - 1$.)

For the top part, choose the parametrization $x = t$ and $y = 2$ for $0 \leq t \leq 4$. We then need the maximum and minimum values of

$$h_2(t) = f(t, 2) = (t-2)^2 + (t-4)^2 - 1 = 2t^2 - 12t + 19$$

for $0 \leq t \leq 4$. Then $h_2'(t) = 4t - 12$, which is zero when $t = 3$. This number is in the interval $[0, 4]$, and $h_2(3) = 1$. At the endpoints we get $h_2(0) = 19$ and $h_2(4) = 3$.

For the left side, choose the parametrization $x = 0$ and $y = t$ for $0 \leq t \leq 2$. We then need the maximum and minimum values of

$$h_3(t) = f(0, t) = (-t)^2 + (-4)^2 - 1 = t^2 + 15$$

for $0 \leq t \leq 2$. Then $h_3'(t) = 2t$, which is zero when $t = 0$. This number is in the interval $[0, 2]$ (in fact, it is an endpoint), and $h_3(0) = 15$. It remains to check the other endpoint. At the endpoint $t = 2$ we get $h_3(2) = 19$. (Actually, these values of t give points in the plane we already considered when looking at the horizontal sides of the boundary.)

For the right side, choose the parametrization $x = 4$ and $y = t$ for $0 \leq t \leq 2$. We then need the maximum and minimum values of

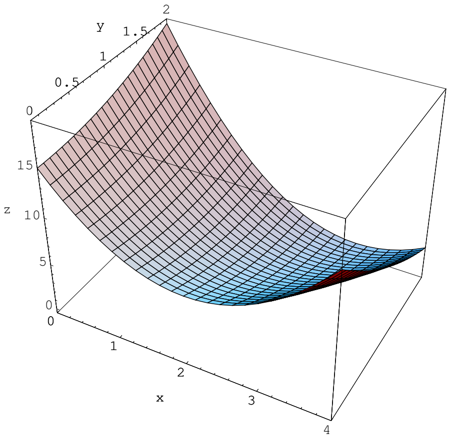
$$h_4(t) = f(4, t) = (4-t)^2 + 0^2 - 1 = (4-t)^2 - 1$$

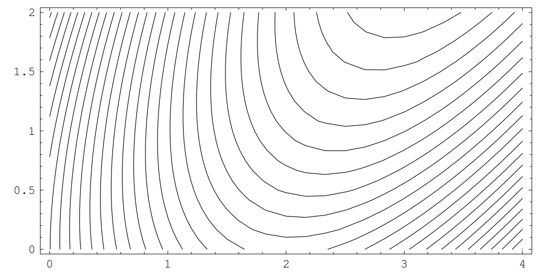
for $0 \leq t \leq 2$. (It is better not to multiply this out.) Then $h_4'(t) = -2(4-t)$, which is zero when $t = 4$. This number is in not the interval $[0, 2]$, so we ignore it. (Again, if you use this number, you will get the *wrong* answer to the problem, because $h_4(4) = f(4, 4) = -1$ is less than any value f has on Y.) It remains to check the endpoints. At the endpoint $t = 0$ we get $h_4(0) = 15$, and at the endpoint $t = 2$ we get $h_4(2) = 3$. (Actually, these values of t give points in the plane we already considered when looking at the horizontal sides of the boundary.)

The largest value we found is 19 and the smallest is 1, so these are the maximum and minimum values of f on Y. □

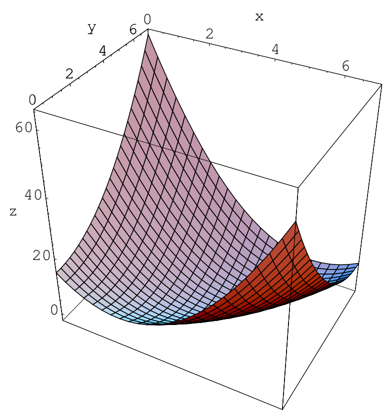
The following graphs are provided only for visualization; they are **not** required as part of the solution.

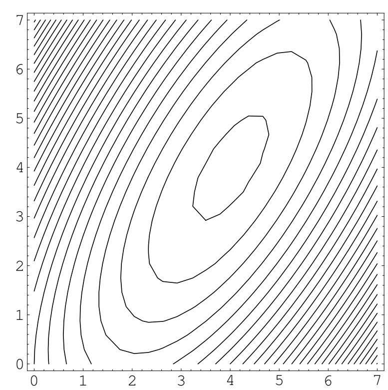
Here are a graph and a contour plot of the function over the region Y:





Here are a graph and a contour plot of the function over the larger region $[0,7]\times[0,7]$. These show the critical point found above that is not in Y.





Problem 11 (12 points)**.** Find an equation for the plane that contains the line $x = 1 - 2t$, $y = 3 + 2t$, $z = t - 1$ and contains the point $(0, 1, 2)$.

Solution. We find two directions in the plane, and take their cross product to get a normal vector.

The plane contains the points $(0, 1, 2)$ (given), $(1,\ 3,\ -1)$ (gotten by taking $t = 0$ in the given parametrization of the line), and $(-1, 5, 0)$ (gotten by taking $t = 1$ in the given parametrization of the line). So I will take

$$\begin{aligned}
\mathbf{n} &= [\langle 1,\ 3,\ -1\rangle - \langle 0, 1, 2\rangle] \times [\langle -1, 5, 0\rangle - \langle 0, 1, 2\rangle] \\
&= \langle 1, 2, -3\rangle \times \langle -1, 4, -2\rangle \\
&= \big\langle (2)(-2) - (-3)(4),\ (-3)(-1) - (1)(-2),\ (1)(4) - (2)(-1)\big\rangle \\
&= \langle 8,\ 5,\ 6\rangle.
\end{aligned}$$

Now the equation of the plane is

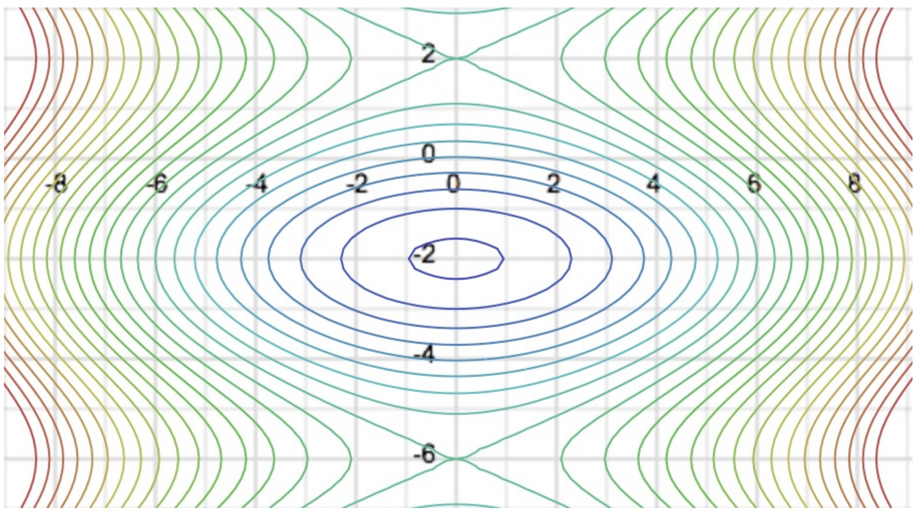
$$\langle 8,\ 5,\ 6\rangle \cdot (\langle x, y, z\rangle - \langle 0, 1, 2\rangle) = 0,$$

which is $8x+5(y-1)+6(z-2) = 0$. This simplifies to $8x+5y+6z-17 = 0$. (The simplification is *required*.) □

Alternate solution. Take the first direction in the plane to be $\langle 1,\ 3,\ -1\rangle - \langle 0, 1, 2\rangle$, as in the first solution, and take the second direction to be a direction vector for the given line, say $\langle -2, 2, 1\rangle$. This gives the same cross product, and the same equation, as in the first solution. □

There are many other possibilities. One simple one is that I could have taken the cross product in the other order, ending up with the equation $-8x - 5y - 6z + 17 = 0$. (It is easy to see that this equation determines the same plane.)

Problem 12 (3 points/part)**.** The picture below is a partial of a contour plot of a function $z = T(x, y)$. The contour lines are evenly spaced, with the darkest red (at the right and left) being at the value 9.8 and the darkest blue (in the middle) being at the value -1.9.



For each of the following questions, give an answer and (this is important) a brief justification.

(1) Is $D_1T(-6,\ -2)$ near zero, clearly positive, or clearly negative?
(2) Is $D_2T(-6,\ -2)$ near zero, clearly positive, or clearly negative?
(3) Is $(-6,\ -2)$ close to being a local minimum, a local maximum, a saddle point, or none of these?
(4) Is $(0,\ -2)$ close to being a local minimum, a local maximum, a saddle point, or none of these?
(5) Give a unit vector $\mathbf{u}$ which points roughly in the same direction as $\mathbf{grad}(T)(6,\ -4)$.

Solution. For convenience, we think of this as a topographic map, and follow the usual map convention, in which right is east and up is north. Thus redder contours are higher and bluer contours are lower.

(1) $D_1T(-6,\ -2) < 0$, since as you go east through $(-6,\ -2)$ you are going away from red, that is, down.
(2) $D_2T(-6,\ -2) \approx 0$, since as you go north through $(-6,\ -2)$, you are going down (towards blue) before you get there, but down (away from blue) after you leave this point.
(3) None of these. The point $(-6,\ -2)$ is not close to being any of a local minimum, a local maximum, or a saddle point, since $D_1T(-6,\ -2)$ isn't even close to zero.
(4) The point $(0,\ -2)$ is close to being a local minimum. It is inside a small region for which, if you leave the region in any direction, you are going towards red, that is, up.

(5) The obvious choice is $\mathbf{u} = \langle 1/\sqrt{2},\ -1/\sqrt{2}\rangle$. The gradient points in the direction of steepest ascent. The vector $\langle 1, -1\rangle$ points roughly in the right direction, since in this direction the function increases (you move towards red) and this vector is roughly orthogonal to the level curve through $(6, -4)$. Multiply by $1/\|\langle 1, -1\rangle\| = 1/\sqrt{2}$ to get a unit vector.

□

Problem 13 (14 points)**.** A drone has acceleration $\mathbf{a}(t) = 2\mathbf{j} + 8\cos(2t)\mathbf{k}$. If it has initial velocity $\mathbf{v}(0) = \mathbf{i} + \mathbf{k}$ and initial position $\mathbf{r}(0) = 2\mathbf{i} + 12\mathbf{k}$, find its velocity and position vectors.

Solution. By integrating, we get

$$\mathbf{v}(t) = 2t\mathbf{j} + 4\sin(2t)\mathbf{k} + \mathbf{C}$$

for some constant $\mathbf{C}$. Since $\mathbf{v}(0) = \mathbf{i} + \mathbf{k}$, we have $\mathbf{C} = \mathbf{i} + \mathbf{k}$. Therefore

$$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + (4\sin(2t) + 1)\mathbf{k}.$$

Integrating again, we get

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + (-2\cos(2t) + t)\mathbf{k} + \mathbf{D}$$

for some constant $\mathbf{D}$. Since $\mathbf{r}(0) = 2\mathbf{i} + 12\mathbf{k}$, we have $\mathbf{D} = 2\mathbf{i} + 14\mathbf{k}$. So

$$\mathbf{r}(t) = (2 + t)\mathbf{i} + t^2\mathbf{j} + (-2\cos(2t) + t + 14)\mathbf{k}.$$

□

Problem 14 (12 points)**.** Find $\lim_{(x,y)\to(0,0)} x^2 \sin\left(\frac{x}{x^2 + y^2}\right)$, giving reasons for your answer, or explain why it does not exist.

Solution. Use the Squeeze Theorem. Clearly $-1 \leq \sin\left(\frac{x}{x^2 + y^2}\right) \leq 1$, and $x^2 \geq 0$, so

$$-x^2 \leq x^2 \sin\left(\frac{x}{x^2 + y^2}\right) \leq x^2$$

for all $(x, y) \neq (0, 0)$. Also,

$$\lim_{(x,y)\to(0,0)} x^2 = \lim_{(x,y)\to(0,0)} (-x^2) = 0,$$

so $\lim_{(x,y)\to(0,0)} x^2 \sin\left(\frac{x}{x^2 + y^2}\right) = 0$. □

Problem 15 (25 extra credit points)**.** Find the maximum and minimum values of the function $f(x, y, z) = 3x^2 + y^2 - y - z^2$ on the solid cylinder bounded by $x^2 + z^2 = 4$, $y = 3$, and $y = -1$. Explain the reasons for your procedure.

Solution. Call the region Y. For the same reason as in two dimensions, if f has a local maximum or minimum at (x_0, y_0, z_0), and if $f_x(x_0, y_0, z_0)$, $f_y(x_0, y_0, z_0)$, and $f_z(x_0, y_0, z_0)$ all exist, then all of these partial derivatives must be zero. For obvious reasons, we will call a point (x_0, y_0, z_0) a *critical point* if

$$f_x(x_0, y_0, z_0) = f_y(x_0, y_0, z_0) = f_z(x_0, y_0, z_0) = 0,$$

or if some of these partial derivatives don't exist and the rest are zero. If the maximum or minimum value of f on Y occurs at a point (x_0, y_0, z_0) in the interior of Y, then (x_0, y_0, z_0) must be a critical point.

So we start by finding the critical points. We calculate:

$$f_x(x, y, z) = 6x, \qquad f_y(x, y, z) = 2y - 1, \qquad \text{and} \qquad f_z(x, y, z) = -2z.$$

These exist everywhere. To find the critical points, we therefore need to solve the simultaneous equations

$$6x = 0, \qquad 2y - 1 = 0, \qquad \text{and} \qquad -2z = 0.$$

There is obviously exactly one solution, namely $x = 0$, $y = \frac{1}{2}$, and $z = 0$. We have $f\left(0, \frac{1}{2}, 0\right) = -\frac{1}{4}$.

The boundary of Y consists of a curved surface forming the side of the cylinder and flat end disks. We will use the methods of two dimensional maximization and minimization problems for each of these, and we will apply them by parametrizing the surfaces.

For the side of the cylinder, choose the parametrization $x = 2\cos(\theta)$, $y = t$, and $z = 2\sin(\theta)$ for $0 \leq \theta \leq 2\pi$ and $-1 \leq t \leq 3$. We then need the maximum and minimum values of

$$g_1(\theta, t) = f(2\cos(\theta),\, t,\, 2\sin(\theta)) = 12\cos^2(\theta) + t^2 - t - 4\sin^2(\theta)$$

for $0 \leq \theta \leq 2\pi$ and $-1 \leq t \leq 3$. It makes things a little simpler to use the identity $\cos^2(\theta) + \sin^2(\theta) = 1$ to rewrite this as

$$g_1(\theta, t) = 12 - 16\sin^2(\theta) + t^2 - t.$$

We calculate

$$(g_1)_\theta(\theta, t) = -32\sin(\theta)\cos(\theta) \qquad \text{and} \qquad (g_1)_t(\theta, t) = 2t - 1.$$

These exist everywhere. To find the critical points, we therefore need to solve the simultaneous equations

$$-32\sin(\theta)\cos(\theta) = 0 \qquad \text{and} \qquad 2t - 1 = 0.$$

The solutions in $[0,\, 2\pi] \times [-1,\, 3]$ are

$$\left(0,\, \tfrac{1}{2}\right), \qquad \left(\tfrac{\pi}{2},\, \tfrac{1}{2}\right), \qquad \left(\pi,\, \tfrac{1}{2}\right), \qquad \left(\tfrac{3\pi}{2},\, \tfrac{1}{2}\right), \qquad \text{and} \qquad \left(2\pi,\, \tfrac{1}{2}\right).$$

We evaluate g_1 at each of these points, getting

$$g_1\left(0,\, \tfrac{1}{2}\right) = 12 - \tfrac{1}{4}, \qquad g_1\left(\tfrac{\pi}{2},\, \tfrac{1}{2}\right) = -4 - \tfrac{1}{4}, \qquad g_1\left(\pi,\, \tfrac{1}{2}\right) = 12 - \tfrac{1}{4},$$
$$g_1\left(\tfrac{3\pi}{2},\, \tfrac{1}{2}\right) = -4 - \tfrac{1}{4}, \qquad \text{and} \qquad g_1\left(2\pi,\, \tfrac{1}{2}\right) = 12 - \tfrac{1}{4}.$$

We are not yet done with finding the extreme values of g_1! We still need to look at the values of g_1 on the boundary of $[0,\, 2\pi] \times [-1,\, 3]$. On the bottom, we consider $h_1(s) = g_1(s,\, -1) = 14 - 16\sin^2(s)$ for $0 \leq \theta \leq 2\pi$. Actually, we know even without calculus that the minimum and maximum values of $\sin^2(s)$ on $[0,\, 2\pi]$ are 0 and 1, so the maximum and minimum values of h_1 on $[0,\, 2\pi]$ are 14 and $12 - 16 = -2$.

Looking at the top of $[0,\, 2\pi] \times [-1,\, 3]$, we get the function $h_2(s) = g_1(s,\, 3) = 18 - 16\sin^2(s)$ for $0 \leq \theta \leq 2\pi$. By the same reasoning as above, the maximum and minimum values of h_1 on $[0,\, 2\pi]$ are 18 and 2.

On the left side, we get $h_3(s) = g_1(0, s) = 12 + s^2 - s$ for $s \in [-1,\, 3]$. We have $h_3'(s) = 2s - 1$, which is zero when $s = \frac{1}{2}$. This number is in the interval $[-1,\, 3]$, and $h_3\left(\frac{1}{2}\right) = 12 - \frac{1}{4}$. We need not look at the endpoints, because these were already checked when considering the top and bottom sides of $[0,\, 2\pi] \times [-1,\, 3]$.

On the right, we look at the function $h_4(s) = g_1(2\pi, s) = 12 + s^2 - s$ for $s \in [-1,\, 3]$. This is exactly the same problem as for the left, so no more work is needed.

We conclude that the maximum and minimum values of f on the curved boundary of the solid are 18 and $-4 - \frac{1}{4}$.

Now we consider the flat end in the positive y direction. We choose the parametrization $x = u$, $y = 3$, and $z = v$ for (u, v) in the closed disk D with center $(0, 0)$ and radius 2.

We then need the maximum and minimum values of

$$g_2(u, v) = f(u, 3, v) = 3u^2 - v^2 + 6$$

for $(u, v) \in D$. We calculate

$$(g_2)_u(u, v) = 6u \qquad \text{and} \qquad (g_1)_v(u, v) = -2v.$$

These exist everywhere. To find the critical points, we therefore need to solve the simultaneous equations

$$6u = 0 \qquad \text{and} \qquad -2v = 0.$$

There is obviously a unique solution, namely $(u, v) = (0, 0)$. This point is in D, and $g_2(0, 0) = 6$.

We should also check the boundary of D, but that was already done when we looked at the side of the cylinder, and need not be repeated.

Finally we consider the flat end in the negative y direction. We choose the parametrization $x = u$, $y = -1$, and $z = v$ for (u, v) in the closed disk D with center $(0, 0)$ and radius 2 (the same set as used for g_2).

We then need the maximum and minimum values of

$$g_3(u, v) = f(u,\, -1,\, v) = 3u^2 - v^2 + 2$$

for $(u, v) \in D$. This function has the same partial derivatives as g_2, and therefore the same critical points, namely $(u, v) = (0, 0)$. This point is in D, and $g_3(0, 0) = 2$. Again, we should check the boundary of D, but that was already done when we looked at the side of the cylinder, and need not be repeated.

The largest value we found is 18, and the smallest is $-4 - \frac{1}{4} = -\frac{17}{4}$, so these are the maximum and minimum values of f on Y. □