SOLUTIONS TO FINAL EXAM SAMPLE PROBLEMS

The final exam is cumulative, but will give higher weight to material not tested before, that is, after the second midterm. (Because of the short period after the second midterm, this will still probably be less than half the exam, perhaps substantially less.)

Most problems will be similar to WeBWorK problems, written homework problems, problems from the real and sample midterms, and the problems here. Note, though, that the exact form of functions which appear could vary substantially. In particular, a problem requiring computation of the limit or derivative of a single variable function could be modified to use any other single variable function; the methods required to find the limit or derivative could be very different. (This includes computations of partial derivatives, which are really just derivatives of single variable functions.)

Be sure to get the notation right! (This is a frequent source of errors.) The right notation will help you get the mathematics right, and incorrect notation will lose points even if there are no other errors.

The problems have point values attached, which give a rough idea of the point values problems requiring a similar amount of work will have on the real exam. The real final will total 200 points, and the problems here total much more than that.

This problem list mostly does not repeat earlier problem types. It contains mostly problems on material after Midterm 2 and problems of types on previous material which should have been in previous sample problem lists but weren’t. You also need to review the earlier sample problem collections!

Problem 1 (8 points). You want to use Lagrange multipliers to minimize the quantity $ye^{x-z}$ subject to the constraints $9x^2 + 4y^2 + 36z^2 = 36$ and $xy = 1 - yz$. Set up, but do not attempt to solve, equations for the points $(x, y, z)$ at which to examine the values of $f$.

Solution. We set

$f(x, y, z) = ye^{x-z} \quad g(x, y, z) = 9x^2 + 4y^2 + 36z^2, \quad \text{and} \quad h(x, y, z) = xy + yz.$

The Lagrange multiplier equations are then

$g(x, y, z) = 36$

$h(x, y, z) = 1$

$D_1 f(x, y, z) = \lambda D_1 g(x, y, z) + \mu D_1 h(x, y, z)$,

$D_2 f(x, y, z) = \lambda D_2 g(x, y, z) + \mu D_2 h(x, y, z)$

$D_3 f(x, y, z) = \lambda D_3 g(x, y, z) + \mu D_3 h(x, y, z)$.
Putting in what the functions are, we get:

\[ 9x^2 + 4y^2 + 36z^2 = 36 \]
\[ xy - yz = 1 \]
\[ ye^{x-z} = \lambda(18x^2) + \mu y \]
\[ e^{x-z} = \lambda(8y^2) + \mu(x - z) \]
\[ -ye^{x-z} = \lambda(72z^2) + \mu y. \]

Be sure to include the constraint equations! The solution isn’t complete without them!

Don’t try to solve the equations!

Problem 2 (18 points). Use Lagrange multipliers to find the maximum and minimum values of \( 2x^2 + 3y^2 - 4x - 5 \) on the circle \( x^2 + y^2 = 16 \).

Solution. We set

\[ f(x, y) = 2x^2 + 3y^2 - 4x - 5 \quad \text{and} \quad g(x, y) = x^2 + y^2. \]

The Lagrange multiplier equations are then

\[ g(x, y) = 16, \quad D_1f(x, y) = \lambda D_1g(x, y), \quad \text{and} \quad D_1f(x, y) = \lambda D_1g(x, y). \]

Putting in what the functions are, we get

\[ x^2 + y^2 = 16, \quad 4x - 4 = \lambda(2x), \quad \text{and} \quad 6y = \lambda(2y). \]

From \( 6y = \lambda(2y) \) we get \( y = 0 \) or \( \lambda = 3 \). If \( y = 0 \) then \( x^2 + y^2 = 16 \) implies \( x = 4 \) or \( x = -4 \), giving us the points \((4, 0)\) and \((-4, 0)\) to consider. If \( \lambda = 3 \), then \( 4x - 4 = 6x \), so \( x = -2 \). The equation \( x^2 + y^2 = 16 \) now implies \( y = \sqrt{12} \) or \( y = -\sqrt{12} \). So we need to consider the points \((-2, \sqrt{12})\) and \((-2, -\sqrt{12})\).

We now evaluate:

\[ f(4, 0) = 11, \quad f(-4, 0) = 43, \quad f(-2, \sqrt{12}) = 47, \quad \text{and} \quad f(-2, -\sqrt{12}) = 47. \]

The largest value is 47, which occurs at \((-2, \sqrt{12})\) and \((-2, -\sqrt{12})\), and the smallest value is 11, which occurs at \((4, 0)\).

Problem 3 (8 points). You want to use Lagrange multipliers to maximize the quantity \( x^4 + y^6 + z^8 \) subject to the constraint \( 2x^2 + 4y^2 = 65 - 6z^2 \). Set up, but do not attempt to solve, equations for the points \((x, y, z)\) at which to examine the values of \( f \).

Solution. We set

\[ f(x, y, z) = x^4 + y^6 + z^8 \quad \text{and} \quad g(x, y, z) = 2x^2 + 4y^2 + 6z^2. \]
The Lagrange multiplier equations are then

\[ g(x, y, z) = 65 \]
\[ D_1 f(x, y, z) = \lambda D_1 g(x, y, z) \]
\[ D_2 f(x, y, z) = \lambda D_2 g(x, y, z) \]
\[ D_3 f(x, y, z) = \lambda D_3 g(x, y, z). \]

Putting in what the functions are, we get

\[ 2x^2 + 4y^2 + 6z^2 = 65 \]
\[ 4x^3 = \lambda(4x) \]
\[ 6y^5 = \lambda(8y) \]
\[ 8x^7 = \lambda(12z). \]

Be sure to include the constraint equations! The solution isn’t complete without them!

Don’t try to solve the equations!

**Problem 4** (8 points). Set \( f(x, y, z) = x \sin(yz^2) \). Find the directional derivative of \( f \) at the point \((1, 3, 0)\) in the (unit length) direction determined by the vector \( 2\mathbf{i} - \mathbf{j} - 3\mathbf{k} \).

**Solution.** We calculate

\[ \text{grad}(f)(x, y, z) = \langle \sin(yz^2), xz^2 \cos(yz^2), 2xyz \cos(yz^2) \rangle. \]

Therefore \( \text{grad}(f)(1, 3, 0) = \langle 0, 0, 0 \rangle \). So the directional derivative of \( f \) at the point \((1, 3, 0)\) in every direction is zero.

Without the fortuitous outcome \( \text{grad}(f)(1, 3, 0) = \langle 0, 0, 0 \rangle \), we would have had to find the appropriate unit vector, which is

\[ \frac{1}{\|2\mathbf{i} - \mathbf{j} - 3\mathbf{k}\|}(2\mathbf{i} - \mathbf{j} - 3\mathbf{k}) = \frac{1}{\sqrt{14}}(2\mathbf{i} - \mathbf{j} - 3\mathbf{k}) = \left\langle \frac{2}{\sqrt{14}}, -\frac{1}{\sqrt{14}}, -\frac{3}{\sqrt{14}} \right\rangle, \]

and then evaluate

\[ \text{grad}(f)(1, 3, 0) \cdot \left\langle \frac{2}{\sqrt{14}}, -\frac{1}{\sqrt{14}}, -\frac{3}{\sqrt{14}} \right\rangle. \]

**Problem 5** (14 points). On the planet Yuggxth, a ball is thrown into the air from the origin towards the east (positive \( x \) direction), with initial velocity \( 50\mathbf{i} + 80\mathbf{k} \) (with speed measured in feet per second). Because of spin on the ball, it experiences a southwards acceleration of \( 4 \text{ feet/second}^2 \). The acceleration due to gravity at the surface of the planet Yuggxth is exactly \( 20 \text{ feet/second}^2 \). Where does the ball hit the ground, and with what speed?

**Solution.** We are given

\[ \mathbf{r}(0) = \langle 0, 0, 0 \rangle, \quad \mathbf{r}'(0) = \langle 50, 0, 80 \rangle, \quad \text{and} \quad \mathbf{r}''(t) = \langle 0, -4, -20 \rangle, \]

the last from \( t = 0 \) until the time the ball hits the ground. Integrating the last equation once gives

\[ \mathbf{r}'(t) = \langle c_1, -4t + c_2, -20t + c_3 \rangle \]
4 SOLUTIONS TO FINAL EXAM SAMPLE PROBLEMS

for constants $c_1$, $c_2$, and $c_3$. Using $r'(0) = \langle 50, 0, 80 \rangle$, we get

$$c_1 = 50, \quad c_2 = 0, \quad \text{and} \quad c_3 = 80.$$ 

So

$$r'(t) = \langle 50, -4t, -20t + 80 \rangle.$$ 

Integrate again:

$$r(t) = \langle 50t + d_1, -2t^2 + d_2, -10t^2 + 5080t + d_3 \rangle$$ 

for constants $d_1$, $d_2$, and $d_3$. Since $r(0) = \langle 0, 0, 0 \rangle$, we get $d_1 = d_2 = d_3 = 0$. So

$$r(t) = \langle 50t, -2t^2, -10t^2 + 80t \rangle.$$ 

The ball hits the ground at the smallest time $t > 0$ such that $-10t^2 + 80t = 0$, which is $t = 8$ seconds. (Units are required.) The position is then

$$r(8) = \langle 50(8), -2(8^2), -10(8^2) + 80(8) \rangle = \langle 400, -128, 0 \rangle \text{ feet}.$$ 

This is 400 feet east and 128 feet south of where it was thrown from. (Units are required.) □

Problem 6 (12 points). Find $\lim_{(x,y) \to (0,0)} (x^2 + y^4) \cos \left( \frac{7}{2x^6 + y^2} \right)$, giving reasons for your answer, or explain why it does not exist.

Solution. Use the Squeeze Theorem. Clearly $-1 \leq \cos \left( \frac{7}{2x^6 + y^2} \right) \leq 1$, and $x^2 + y^4 \geq 0$, so

$$-(x^2 + y^4) \leq (x^2 + y^4) \cos \left( \frac{7}{2x^6 + y^2} \right) \leq x^2 + y^4$$

for all $(x, y) \neq (0, 0)$. Also,

$$\lim_{(x,y) \to (0,0)} (x^2 + y^4) = \lim_{(x,y) \to (0,0)} \left[ -(x^2 + y^4) \right] = 0,$$

so $\lim_{(x,y) \to (0,0)} (x^2 + y^4) \cos \left( \frac{7}{2x^6 + y^2} \right) = 0$. □

Problem 7 (10 points). The picture below is a partial of a contour plot of a function $z = Q(x, y)$. The contour lines are evenly spaced, with the darkest red (at the top and bottom) being at the value 4 and the darkest blue (at the right) being at the value $-4$. 

![Contour Plot Image]
Identify all critical points in the region shown, and for each one say whether it is a local minimum, local maximum, saddle point, or none of these. Give reasons.

Solution. For convenience, we think of this as a topographic map, and follow the usual map convention, in which right is east and up is north. Thus redder contours are higher and bluer contours are lower.

There is a local maximum or minimum inside each of the closed loop contour lines with no other contour lines inside. Thus, there is a local maximum or minimum at each of the points \((-1, 1), (1, -1), \) and \((1, 0)\). As you go north along the line \(x = -1\), you go down until you are somewhere near \((-1, -1)\), up until you are somewhere near \((-1, 0)\), down until you are somewhere near \((-1, 1)\), and then up. So there are local minimums near \((-1, 1)\) and \((-1, -1)\). As you go west along the \(x\)-axis, you go up until you are somewhere near \((1, 0)\), and then you start going down again. So there is a local maximum near \((-1, -1)(1, 0)\).
There are saddle points at \((-1, 0), (1, 1),\) and \((1, -1).\) As you go through \((-1, 0)\) going north, you are going up until you get there and going down after you pass this point. As you go through \((-1, 0)\) going east, you are going down until you get there and going up after you pass this point. So there is a pass here. As you go through either of \((1, 1),\) and \((1, -1)\) going north, you are going down until you get there and going up after you pass this point. As you go through either of \((1, 1),\) and \((1, -1)\) going east, you are going up until you get there and going down after you pass this point. So there is a pass at each of these two points.

**Problem 8** (8 points). A function \(f\) of two variables satisfies \(f(2, 7) = -3, f_x(2, 7) = 0,\) \(f_y(2, 7) = 0, f_{xx}(2, 7) = -4, f_{yy}(2, 7) = -5,\) and \(f_{xy}(2, 7) = 2.\) Does \(f\) have a local minimum at \((2, 7),\) a local maximum at \((2, 7),\) a saddle point at \((2, 7),\) none of these, or is it impossible to determine? Why?

**Solution.** The function \(f\) has a critical point at \((2, 7),\) because \(f_x(2, 7) = f_y(2, 7) = 0.\) Next, \(D = f_{xx}(2, 7)f_{yy}(2, 7) - [f_{xy}(2, 7)]^2 = (-4)(-5) - 2^2 = 16 > 0,\) so \(f\) has a local extremum at \((2, 7).\) Finally, \(f_{xx}(2, 7) < 0,\) so it is a local maximum.

**Problem 9** (15 points). For the function \(f(x, y) = 4y^3 - 3y + 12x^2y,\) find all critical points and identify each as a local maximum, local minimum, or saddle point.

**Solution.** We calculate:

\[
\begin{align*}
  f_x(x, y) &= 24xy \\
  f_y(x, y) &= 12y^2 - 3 + 12x^2.
\end{align*}
\]

These exist everywhere. To find the critical points, we therefore need to solve the simultaneous equations

\[
\begin{align*}
  24xy &= 0 \\
  12y^2 - 3 + 12x^2 &= 0,
\end{align*}
\]
equivalently

\[
\begin{align*}
  xy &= 0 \\
  4x^2 + 4y^2 &= 1.
\end{align*}
\]

The first equation gives \(x = 0\) or \(y = 0.\) If \(x = 0\) then the second equation gives \(y = \pm \frac{1}{2},\) and if \(y = 0\) then the second equation gives \(x = \pm \frac{1}{2}.\) Thus, \((0, \frac{1}{2}),\) \((0, -\frac{1}{2}),\) \((\frac{1}{2}, 0),\) and \((-\frac{1}{2}, 0)\) are all possible critical points. One checks that both partial derivatives really do vanish at each of these points.

We now apply the second derivative test. We have

\[
\begin{align*}
  f_{xx}(x, y) &= 24y, \\
  f_{yy}(x, y) &= 24, \\
  f_{xy}(x, y) &= 24x.
\end{align*}
\]

So \(D(0, \frac{1}{2}) = (12)(12) - (0)^2 = 144 > 0\) and \(f_{xx}(0, \frac{1}{2}) = 12 > 0,\) so \(f\) has a local minimum at \((0, \frac{1}{2}).\) Also, \(D(0, -\frac{1}{2}) = (-12)(-12) - (0)^2 = 144 > 0\) and \(f_{xx}(0, -\frac{1}{2}) = -12 < 0,\) so \(f\) has a local maximum at \((0, -\frac{1}{2}).\) Next, \(D\left(\frac{1}{2}, 0\right) = (0)(0) - 12^2 = -144 < 0,\) and \(f\) has a saddle point at \((\frac{1}{2}, 0).\) Finally, \(D\left(-\frac{1}{2}, 0\right) = (0)(0) - (-12)^2 = -144 < 0,\) and \(f\) has another saddle point at \((-\frac{1}{2}, 0).\) 

**Problem 10** (10 points). Suppose the position of a particle at time \(t\) is given by \(\mathbf{r}(t) = ti + t^2\mathbf{j} + 3tk.\) Find the tangential and normal components of its acceleration vector.

**Solution.** Begin by computing \(\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 3\mathbf{k}.\) So

\[
\|\mathbf{r}'(t)\| = \sqrt{1 + (2t)^2 + 9} = \sqrt{10 + 4t^2}.
\]
The tangential component of the acceleration vector is the derivative of this, which is
\[ 4t(10 + 4t^2)^{-1/2}. \]

The normal component is
\[ \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|}. \]
(This came from the formula involving curvature.) So calculate
\[ r''(t) = 2j, \quad r'(t) \times r''(t) = -6i + 2k, \quad \text{and} \quad \|r'(t) \times r''(t)\| = 2\sqrt{10}. \]
Thus the normal component is
\[ \frac{2\sqrt{10}}{\sqrt{10 + 4t^2}}. \]

**Problem 11** (10 points). Find the velocity and position vectors of a particle that has acceleration
\[ a(t) = -tk, \] initial velocity \( v(0) = i + j - k, \) and initial position \( r(0) = 2i + 3j. \)

*Solution.* By integrating, we get
\[ v(t) = \frac{-1}{2}t^2k + C \]
for some constant \( C. \) Since \( v(0) = i + j - k, \) we have \( C = i + j - k. \) Therefore
\[ v(t) = i + j - (1 + \frac{1}{2}t^2)k. \]

Integrating again, we get
\[ r(t) = ti + tj - (t + \frac{1}{6}t^3)k + D \]
for some constant \( D. \) Since \( r(0) = 2i + 3j, \) we have \( D = 2i + 3j. \) So
\[ r(t) = (2 + t)i + (3 + t)j - (t + \frac{1}{6}t^3)k. \]

**Problem 12** (6 points/part). Let \( r(t) = (t^2, 2t, \ln(t)). \)

1. Find the unit tangent vector.
2. Find the the unit normal vector.
3. Find the the binormal vector.
4. Find the the curvature.

*Solution to Part (1).* The tangent vector is \( r'(t) = (2t, 2, 1/t). \) To get the unit tangent vector, multiply by 1 divided by the length:
\[ |r'(t)| = \sqrt{4t^2 + 4 + 1/t^2} = \sqrt{(2t + 1/t)^2} = |2t + 1/. |.
\]

So
\[ T(t) = \left( \frac{1}{|r'(t)|} \right) r'(t) = \left( \frac{2t}{|2t + 1/t|}, \frac{2}{|2t + 1/t|}, \frac{1/t}{|2t + 1/t|} \right). \]

Further simplification depends on the sign of \( t. \) For use in the remaining parts, we carry it out. If \( t > 0, \) then \( 2t + 1/t > 0, \) so
\[ T(t) = \left( \frac{2t}{2t + 1/t}, \frac{2}{2t + 1/t}, \frac{1/t}{2t + 1/t} \right) = \left( \frac{2t^2}{2t^2 + 1}, \frac{2t}{2t^2 + 1}, \frac{1}{2t^2 + 1} \right) = \left( \frac{1}{2t^2 + 1} \right) (2t^2, 2t, 1). \]
If \( t < 0 \), then \(|2t + 1/t| = -(2t + 1/t)\), so similarly
\[
T(t) = -\left( \frac{1}{2t^2 + 1} \right) \langle 2t^2, 2t, 1 \rangle.
\]

\[\square\]

**Solution to Part (2).** For \( t > 0 \), use the product rule for scalar multiplication on the last expression for \( T(t) \) to get
\[
T'(t) = -\left( \frac{4t}{(2t^2 + 1)^2} \right) \langle 2t^2, 2t, 1 \rangle + \left( \frac{1}{2t^2 + 1} \right) \langle 4t, 2, 0 \rangle.
\]
Combine terms:
\[
T'(t) = \left( \frac{1}{(2t^2 + 1)^2} \right) \langle -8t^3, -8t^2, -4t \rangle + \left( \frac{1}{2t^2 + 1} \right) \langle 4t(2t^2 + 1), 2(2t^2 + 1), 0 \rangle
\]
\[
= \left( \frac{1}{(2t^2 + 1)^2} \right) \langle 4t, -4t^2 + 2, -4t \rangle = \left( \frac{2}{(2t^2 + 1)^2} \right) \langle 2t, -2t^2 + 1, -2t \rangle.
\]
We want the unit vector in the same direction. Since the factor in front is positive, this is the same as the unit vector in the direction \( \langle 2t, -2t^2 + 1, -2t \rangle \). Now
\[
|\langle 2t, -2t^2 + 1, -2t \rangle| = \sqrt{4t^2 + (4t^4 - 4t^2 + 1) + 4t^4} = \sqrt{4t^4 + 4t^2 + 1} = 2t^2 + 1.
\]
Therefore
\[
N(t) = \left( \frac{1}{2t^2 + 1} \right) \langle 2t, -2t^2 + 1, -2t \rangle.
\]
For \( t < 0 \), the extra minus sign carries through, giving
\[
N(t) = -\left( \frac{1}{2t^2 + 1} \right) \langle 2t, -2t^2 + 1, -2t \rangle.
\]
\[\square\]

**Solution to Part (3).** For \( t > 0 \),
\[
B(t) = T(t) \times N(t) = \left( \frac{1}{2t^2 + 1} \right) \left( \frac{1}{2t^2 + 1} \right) \langle 2t^2, 2t, 1 \rangle \times \langle 2t, -2t^2 + 1, -2t \rangle
\]
\[
= \left( \frac{1}{(2t^2 + 1)^2} \right) \langle -4t^2 - (-2t^2 + 1), -(-4t^3 - 2t), (-4t^4 + 2t^2) - 4t^2 \rangle
\]
\[
= \left( \frac{1}{(2t^2 + 1)^2} \right) \langle -2t^2 + 1, 4t^3 + 2t, -4t^4 - 2t^2 \rangle = \left( \frac{1}{2t^2 + 1} \right) \langle -1, 2t, -2t^2 \rangle.
\]
The answer is the same for \( t < 0 \), because the minus signs cancel. \[\square\]

**Solution to Part (4).** Given the work done already, it is easiest to use the formula
\[
\kappa(t) = \frac{|T'(t)|}{|r'(t)|}.
\]
Substituting results from above (and using a previously done calculation at the second step), for \( t > 0 \) this gives

\[
\kappa(t) = \frac{\langle 2t, -2t^2 + 1, -2t \rangle}{2t + 1/t} = \frac{\left( \frac{2}{(2t^2 + 1)^{1/2}} \right) \left( 2t^2 + 1 \right)}{2t + 1/t} = \frac{2t}{(2t^2 + 1)^2}.
\]

For \( t < 0 \), the denominator in the first fraction is \(-\frac{2t + 1/t}{t}\) instead, so

\[
\kappa(t) = -\frac{2t}{(2t^2 + 1)^2}.
\]

□

**Problem 13** (10 points).

Let \( f \) be the function of two variables given by \( f(x, y) = \sqrt{16 - 4x^2 - y^2} \).

(1) Sketch the domain of \( f \). If the domain has a boundary, be sure to make it clear which parts of the boundary are in the domain and which are not. Give reasons.

(2) Sketch the level curve of this function corresponding to the value \( \sqrt{12} \).

**Solution to Part (1). Incomplete: pictures missing!** The domain consists of all pairs \((x, y)\) in \( \mathbb{R}^2 \) at which the argument of the square root is nonnegative, that is, \( 16 - 4x^2 - y^2 \geq 0 \). Written in a nicer form, it is

\[
\left\{ (x, y): 4x^2 + y^2 \leq 16 \right\}.
\]

This is the inside of the ellipse \( 4x^2 + y^2 \leq 16 \), including the boundary. To graph it, rewrite it as

\[
\frac{x^2}{2^2} + \frac{y^2}{4^2} = 1.
\]

This is the ellipse with center \((0, 0)\) and vertices at \((2, 0), (-2, 0), (0, 4),\) and \((0, -4)\).

□

It is not correct to write \( \left\{ (x, y): 4x^2 + y^2 \leq 16 \right\} \). This formula is wrong, because it tries to describe a subset of \( \mathbb{R} \) rather than a subset of \( \mathbb{R}^2 \).

**Solution to Part (2). Incomplete: pictures missing!** The level curve of all pairs \((x, y)\) in \( \mathbb{R}^2 \) such that \( f(x, y) = \sqrt{12} \). Since \( f(x, y) \geq 0 \) for all \((x, y)\) in the domain of \( f \), this is the same as \( 16 - 4x^2 - y^2 = 12 \), that is, \( 4x^2 + y^2 = 4 \), or

\[
x^2 + \frac{y^2}{2^2} = 1.
\]

This is the ellipse with center \((0, 0)\) and vertices at \((1, 0), (-1, 0), (0, 2),\) and \((0, -2)\).

□

**Problem 14** (20 points). Find the maximum and minimum values of the function \( f(x, y) = -\frac{1}{2}x^2 + y^2 - 2y \) on the closed region bounded by the \( x \)-axis and the graph of \( y = 1 - x^2 \).

**Solution.** Call the region \( D \). Here is a picture of it:
We start by finding the critical points. We calculate:

\[ f_x(x, y) = -x \quad \text{and} \quad f_y(x, y) = 2y - 2. \]

These exist everywhere. To find the critical points, we therefore need to solve the simultaneous equations

\[ -x = 0 \quad \text{and} \quad 2y - 2 = 0. \]

Clearly there is a unique solution, namely \( x = 0 \) and \( y = 1 \). The point (0, 1) is in \( D \), and \( f(0, 1) = -1 \).

The boundary of \( D \) consists of two curves, a lower boundary consisting of the \( x \)-axis for \(-1 \leq x \leq 1\), and an upper boundary consisting of the graph of \( y = 1 - x^2 \) for \(-1 \leq x \leq 1\). (The numbers \(-1 \) and \( 1 \) are the values of \( x \) at which these two curves cross.)

For the lower boundary, choose the parametrization \( x = t \) and \( y = 0 \) for \(-1 \leq x \leq 1\). We then need the maximum and minimum values of \( h_1(t) = f(t, 0) = -\frac{1}{2}t^2 \) for \(-1 \leq x \leq 1\). Even without calculus, these are obvious: 0 at \( t = 0 \), and \(-\frac{1}{2} \) at \( t = \pm 1 \).

For the upper boundary, choose the parametrization \( x = t \) and \( y = 1 - t^2 \) for \(-1 \leq x \leq 1\). Thus, we need the maximum and minimum values of \( h_2(t) = f(t, 1 - t^2) \) for \(-1 \leq x \leq 1\). We calculate:

\[ h_2(t) = -\frac{1}{2}t^2 + (1 - t^2)^2 - 2(1 - t^2) = t^4 - \frac{1}{2}t^2 - 1. \]

Then \( h'_2(t) = 4t^3 - t = (4t^2 - 1)t \), which is zero when \( t = 0 \) and when \( t = \pm \frac{1}{2} \). All these are in the interval \([-1, 1]\). At \( t = 0 \) we get the value \( h_2(2) = -1 \). (Actually, \( t = 0 \) gives the critical point for \( f \) we already found.) At both \( t = \frac{1}{2} \) and \( t = -\frac{1}{2} \) we get \( h_2(t) = \frac{1}{16} - \frac{1}{8} - 1 = -\frac{17}{16} \). Finally, at the endpoints \( t = \pm 1 \) we get \( h_2(t) = -\frac{1}{2} \). (Actually, these values of \( t \) give points in the plane we already considered when looking at the lower boundary.)

The largest value we found was 0 and the smallest was \(-\frac{17}{16}\), so these are the maximum and minimum values of \( f \) on \( D \). \( \square \)

**Problem 15** (20 points). Find the maximum and minimum values of the function \( f(x, y) = (x + y)^4 - \cos(x - y) \) on the closed region bounded by the lines \( x + y = 1 \), \( x - y = 1 \), \( x + y = -1 \), and \( x - y = -1 \).

**Solution.** Call the region \( W \). Here is a picture:
We start by finding the critical points. We calculate:

\[ f_x(x, y) = 4(x + y)^3 + \sin(x - y) \quad \text{and} \quad f_y(x, y) = 4(x + y)^3 - \sin(x - y). \]

(Do not multiply out \((x + y)^4\) or \(4(x + y)^3\). You will only succeed in making the algebra much harder.) These exist everywhere. To find the critical points, we therefore need to solve the simultaneous equations

\[ 4(x + y)^3 + \sin(x - y) = 0 \quad \text{and} \quad 4(x + y)^3 - \sin(x - y) = 0. \]

Adding these two equations gives \(8(x+y)^3 = 0\), so \(x+y = 0\), so \(y = -x\). Subtracting the second equation from the first gives \(2\sin(x-y) = 0\). Substituting \(y = -x\) turns this into \(2\sin(2x) = 0\). The solutions are integer multiples of \(\pi/2\). Therefore the critical points are \((0,0), (\pi/2, -\pi/2), (\pi, -\pi), (3\pi/2, -3\pi/2), \text{etc.}\) as well as \((-\pi/2, \pi/2), (-\pi, \pi), (-3\pi/2, 3\pi/2), \text{etc.}\) Of these, only \((0,0)\) is in the region \(W\). (All others give either \(x - y \geq \pi\) or \(x - y \leq -\pi < -1\.) We therefore only consider \(f(0,0) = -1\).

The boundary of \(W\) consists of four diagonal line segments, one in each quadrant.

For the part in the first quadrant, choose the parametrization \(x = t\) and \(y = 1-t\) for \(0 \leq t \leq 1\). We then need the maximum and minimum values of

\[ h_1(t) = f(t, 1-t) = [t + (1-t)]^4 - \cos(t - (1-t)) = 1 - \cos(2t-1) \]

for \(0 \leq t \leq 1\). Then \(h_1'(t) = 2\sin(2t-1)\), which is zero when \(2t - 1\) is an integer multiple of \(\pi\), that is, \(t = \frac{1}{2}, t = \frac{1}{2} + \frac{\pi}{2}, t = \frac{1}{2} + \pi,\) etc., or \(t = \frac{1}{2} - \frac{\pi}{2}, t = \frac{1}{2} - \pi,\) etc. Of these, only \(\frac{1}{2}\) is in the interval \([0,1]\), so we need only consider \(h_1\left(\frac{1}{2}\right) = 0\). It remains to check the endpoints. At the endpoint \(t = 0\) we get \(h_1(0) = 1 - \cos(-1) = 1 - \cos(1)\), and at the endpoint \(t = 1\) we get \(h_1(1) = 1 - \cos(1)\).

For the part in the second quadrant, choose the parametrization \(x = t\) and \(y = 1+t\) for \(-1 \leq t \leq 0\). We then need the maximum and minimum values of

\[ h_2(t) = f(t, 1+t) = [t + (1+t)]^4 - \cos(t - (1-t)) = (2t+1)^4 - \cos(-1) = (2t+1)^4 - \cos(1) \]

for \(-1 \leq t \leq 0\). Then \(h_2'(t) = 8(2t+1)^3\), which is zero when \(2t + 1 = 0\), that is, when \(t = -\frac{1}{2}\). The number \(-\frac{1}{2}\) is in the interval \([-1, 0]\), and \(h_2\left(-\frac{1}{2}\right) = -\cos(1)\). It remains to check the endpoints. At the endpoint \(t = 0\) we get \(h_2(0) = 1 - \cos(1)\), and at the endpoint \(t = -1\) we get \(h_2(-1) = (-1)^4 - \cos(1) = 1 - \cos(1)\).
For the part in the third quadrant, choose the parametrization \( x = t \) and \( y = -1 - t \) for \(-1 \leq t \leq 0\). We then need the maximum and minimum values of

\[
h_3(t) = f(t, -1-t) = [t + (-1 - t)]^4 - \cos(t - (-1 - t)) = (-1)^4 - \cos(2t + 1) = 1 - \cos(2t + 1)
\]

for \(-1 \leq t \leq 0\). Then \( h_3'(t) = 2 \sin(2t + 1) \), which is zero when \( 2t + 1 \) is an integer multiple of \( \pi \), that is, \( t = -\frac{1}{2}, t = -\frac{1}{2} + \frac{\pi}{2}, t = -\frac{1}{2} + \pi, \) etc., or \( t = -\frac{1}{2} - \frac{\pi}{2}, t = -\frac{1}{2} - \pi, \) etc. Of these, only \(-\frac{1}{2}\) is in the interval \([-1, 0]\), so we need only consider \( h_3\left(-\frac{1}{2}\right) = 0 \). It remains to check the endpoints. At the endpoint \( t = 0 \) we get \( h_3(0) = 1 - \cos(1) \), and at the endpoint \( t = -1 \) we get \( h_3(-1) = 1 - \cos(-1) = 1 - \cos(1) \).

(One can avoid most of this by checking that the values of \( f \) on this part of the boundary are the same as those on the part in the first quadrant.)

For the part in the fourth quadrant, choose the parametrization \( x = t \) and \( y = t - 1 \) for \( 0 \leq t \leq 1 \). We then need the maximum and minimum values of

\[
h_4(t) = f(t, t - 1) = [t + (t - 1)]^4 - \cos(t - (t - 1)) = (2t - 1)^4 - \cos(1)
\]

for \( 0 \leq t \leq 1 \). Then \( h_4'(t) = 8(2t - 1)^3 \), which is zero when \( 2t - 1 = 0 \), that is, when \( t = \frac{1}{2} \). The number \( \frac{1}{2} \) is in the interval \([0, 1]\), and \( h_4\left(\frac{1}{2}\right) = -\cos(1) \). It remains to check the endpoints. At the endpoint \( t = 0 \) we get \( h_4(0) = (-1)^4 - \cos(1) = 1 - \cos(1) \), and at the endpoint \( t = 1 \) we get \( h_4(1) = (-1)^4 - \cos(1) = 1 - \cos(1) \).

(Again, one can avoid most of this by checking that the values of \( f \) on this part of the boundary are the same as those on the part in the second quadrant.)

The values we found are 0, \(-1\), \(-\cos(1)\), and \(1 - \cos(1)\). Since \( 0 < 1 < \pi/2 \), we have \( 0 < \cos(1) < 1 \). Therefore the largest of these is \( 1 - \cos(1) \) and the smallest is \(-1\). \(\square\)

Here is a graph of the function over the region \( W \). It is provided only for visualization; it is \textbf{not} required as part of the solution. To help show where \( W \) is, the graph of \( g(x, y) = -1 \) is also shown over \( W \).
Remark: Any solution which goes wrong by failing to look for possible maximums and minimums in the middle of the boundary curves will lose many points, even if no other errors are made.

Problem 16 (20 points). Find the maximum and minimum values of the function \( f(x, y) = (x + y)^2 + (x - 2)^2 \) on the closed rectangle with vertices \((0, 0), (0, 2), (3, 0), \) and \((3, 2)\).

Solution. Call the region \( Y \). Here is a picture:

We start by finding the critical points. We calculate:
\[
\begin{align*}
    f_x(x, y) &= 2(x + y) \\
    f_y(x, y) &= 2(x + y) + 2(x - 2).
\end{align*}
\]
(You are better off not multiplying out either the formula for the function or the formulas for these partial derivatives.) These exist everywhere. To find the critical points, we therefore need to solve the simultaneous equations
\[
2(x + y) = 0 \quad \text{and} \quad 2(x + y) + 2(x - 2) = 0.
\]
The first equation implies \( y = -x \), and the second equation then implies \( x = 2 \). So there is a unique solution, namely \( x = 2 \) and \( y = -2 \). The point \((2, -2)\) is not in \( Y \), so we ignore it. (If you use this point, you will get the wrong answer to the problem, because \( f(2, -2) = 0 \) is less than any value \( f \) has on \( Y \).)
The boundary of $Y$ consists of four straight line segments, namely the $x$-axis for $0 \leq x \leq 3$ (the bottom), the line $y = 2$ for $0 \leq x \leq 3$ (the top), the $y$-axis for $0 \leq y \leq 2$ (the left side), and the line $x = 3$ for $0 \leq y \leq 2$ (the right side).

For the bottom part, choose the parametrization $x = t$ and $y = 0$ for $0 \leq t \leq 3$. We then need the maximum and minimum values of

$$h_1(t) = f(t, 0) = t^2 + (t - 2)^2 = 2t^2 - 4t + 4$$

for $0 \leq t \leq 3$. Then $h_1'(t) = 4t - 4$, which is zero when $t = 1$. This number is in the interval $[0, 3]$, and $h_1(1) = 2$. (It is easiest to calculate this using the formula $h_1(t) = t^2 + (t - 2)^2$.) It remains to check the endpoints. At the endpoint $t = 0$ we get $h_1(0) = 4$, and at the endpoint $t = 3$ we get $h_1(3) = 10$. (Again, it is easiest to calculate $h_1(3)$ using the formula $h_1(t) = t^2 + (t - 2)^2$.)

For the top part, choose the parametrization $x = t$ and $y = 2$ for $0 \leq t \leq 3$. We then need the maximum and minimum values of

$$h_2(t) = f(t, 2) = (t + 2)^2 + (t - 2)^2 = 2t^2 + 8$$

for $0 \leq t \leq 3$. Then $h_2'(t) = 4t$, which is zero when $t = 0$. This number is in the interval $[0, 3]$ (in fact, it is an endpoint), and $h_2(0) = 8$. It remains to check the other endpoint. At the endpoint $t = 3$ we get $h_2(3) = 26$.

For the left side, choose the parametrization $x = 0$ and $y = t$ for $0 \leq t \leq 2$. We then need the maximum and minimum values of

$$h_3(t) = f(0, t) = t^2 + (-2)^2 = t^2 + 4$$

for $0 \leq t \leq 2$. Then $h_3'(t) = 2t$, which is zero when $t = 0$. This number is in the interval $[0, 2]$ (in fact, it is an endpoint), and $h_3(0) = 4$. It remains to check the other endpoint. At the endpoint $t = 2$ we get $h_3(2) = 8$. (Actually, these values of $t$ give points in the plane we already considered when looking at the horizontal sides of the boundary.)

For the right side, choose the parametrization $x = 3$ and $y = t$ for $0 \leq t \leq 2$. We then need the maximum and minimum values of

$$h_4(t) = f(3, t) = (3 + t)^2 + 1$$

for $0 \leq t \leq 2$. (It is better not to multiply this out.) Then $h_4'(t) = 2(3 + t)$, which is zero when $t = -3$. This number is in not the interval $[0, 2]$, so we ignore it. (Again, if you use this number, you will get the wrong answer to the problem, because $h_4(-3) = f(3, -3) = 1$ is less than any value $f$ has on $Y$.) It remains to check the endpoints. At the endpoint $t = 0$ we get $h_4(0) = 10$, and at the endpoint $t = 2$ we get $h_4(2) = 26$. (Actually, these values of $t$ give points in the plane we already considered when looking at the horizontal sides of the boundary.)

The largest value we found is 26 and the smallest is 2, so these are the maximum and minimum values of $f$ on $Y$. \[\square\]

The following graphs are provided only for visualization; they are not required as part of the solution.

Here are a graph and a contour plot of the function over the region $Y$:
Here are a graph and a contour plot of the function over the larger region $[-2, 5] \times [-5, 2]$. These show the critical point found above that is not in $Y$.

Problem 17 (9 points). Let $h$ be a function such that $h'(t) = \sin(t^3 - t)$. Suppose $f(x, y, z) = x^2 - xyz + xh(y - z)$. Find $D_3 f(x, y, z)$.

Solution. We have

$$D_3 f(x, y, z) = \frac{\partial}{\partial z}(x^2) - \frac{\partial}{\partial z}(xyz) + \frac{\partial}{\partial z}(xh(y - z))$$

$$= 0 - xy + xh'(y - z) \frac{\partial}{\partial z}(y - z)$$

$$= -xy + x \sin((y - z)^3 - (y - z))(-1) = -xy - x \sin((y - z)^3 - y + z).$$

□

Problem 18 (16 points). Let $f(x, y, z) = xy^2 z^3 + \sin(z) - x$. Find an equation for the tangent plane to the level surface $f(x, y, z) = -2$ at the point $(2, 1, 0)$.

Solution. For the normal vector to the plane, we take $n = \text{grad}(f)(2, 1, 0)$. So begin by calculating

$$\text{grad}(f)(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle = \langle y^2 z^3 - 1, 2xyz^3, 3xy^2 z^2 + \cos(z) \rangle$$
and

\[ \text{grad}(f)(2, 1, 0) = (-1, 0, 1). \]

The plane goes through the point \((2, 1, 0)\), so we get the equation

\[ (-1, 0, 1) \cdot ((x, y, z) - (2, 1, 0)) = 0, \]

that is, \(-x + 2 + z = 0\). We simplify this to \(2 - x + z = 0\), or to \(x - z = 2\).

**Problem 19** (6 points). Find the volume of the parallelepiped determined by the vectors

\[ a = i + j - k, \quad b = i - j + k, \quad \text{and} \quad c = -i + j + k. \]

**Solution.** It is

\[ |a \cdot (b \times c)| = |(i + j - k) \cdot (-2i - 2j)| = | -4 | = 4. \]

(Note that \(-4\) is not correct. The volume can't be negative.)

**Problem 20** (2 points). Is the expression

\[ \langle 2, -3, -18 \rangle \times \langle -1, 5, -2 \rangle \times \langle -7, 12, -27 \rangle \]

a scalar, a vector, or not defined?

**Solution.** Not defined, because it is ambiguous. In this case,

\[ ((2, -3, -18) \times (-1, 5, -2)) \times (-7, 12, -27) = (-678, 2543, 1306) \]

but

\[ (2, -3, -18) \times (-1, 5, -2) \times (-7, 12, -27) = (-303, 1952, -359) \].

**Problem 21** (10 points). Consider the curve \(x(t) = t, y(t) = t^2, z(t) = t^3\). Find an equation for the normal plane at the point \((1, 1, 1)\).

**Solution.** The normal plane is orthogonal to the tangent vector. Accordingly, set \(r(t) = \langle t, t^2, t^3 \rangle\). We have \(r(t) = (1, 1, 1)\) when \(t = 1\). So we calculate \(r'(1) = (1, 2, 3)\). Now the equation of the plane is

\[ (1, 2, 3) \cdot ((x, y, z) - (1, 1, 1)) = 0, \]

which is \((x - 1) + 2(y - 1) + 3(z - 1) = 0\). This simplifies to \(x + 2y + 3z - 6 = 0\). (The simplification is required.)

**Problem 22** (12 points). The helix \(r(t) = \cos(t)i + \sin(t)j + tk\) intersects the curve \(s(t) = (1 + t)i + t^2j + t^3k\) at the point \((1, 0, 0)\). Find the angle of intersection of the curves.

**Solution.** We will get the cosine of the angle using the dot product of the tangent vectors.

To start, \(r(t) = (1, 0, 0)\) when \(t = 0\). We have \(r'(t) = -\sin(t)i + \cos(t)j + k\), so \(r'(0) = j + k\). Also, \(s(t) = (1, 0, 0)\) when \(t = 0\). We have \(s'(t) = i + 2tj + 3t^2k\), so \(s'(0) = i\). Therefore \(r'(0) \cdot s'(0) = 0\). The curves thus cross at right angles, and the angle is \(\frac{\pi}{2}\).