MATH 281 (PHILLIPS): SOLUTIONS TO MIDTERM 1

Problem 1 (1 point)**.** True or false: The behavior of cross products is bizarre and horrible.

Solution. Well, I don't think a physicist would say that, but many other people would. □

Problem 2 (2 points per part)**.** In each of the following cases, decide whether the given expression is a scalar, a vector, or not defined.

(1) $(\mathbf{j} - \mathbf{u}) \cdot (\mathbf{v} + 2\mathbf{w})$, if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathbb{R}^3$.

Solution. It is a scalar. Since $\mathbf{j}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^3$, so is their difference. Also, $\mathbf{v}$ and $2\mathbf{w}$ are vectors in $\mathbb{R}^3$, so $\mathbf{v} + 2\mathbf{w}$ is as well. Now the dot product of two vectors in $\mathbb{R}^3$ is defined and is a scalar. □

(2) $\langle 2, -3, -7 \rangle \times \langle -1, 5, -2 \rangle \times \langle -7, 3, -3 \rangle$

Solution. Not defined, because it is ambiguous. □

Remark. In this case, the two versions really are different:

$$\big(\langle 2, -3, -7 \rangle \times \langle -1, 5, -2 \rangle \big) \times \langle -7, 3, -3 \rangle = \langle -54, 74, 200 \rangle$$

and

$$\langle 2, -3, -7 \rangle \times \big(\langle -1, 5, -2 \rangle \times \langle -7, 3, -3 \rangle \big) = \langle -19, -1, -5 \rangle.$$

But you don't need to do this calculation.

If you really want to check, all you need to do is calculate the first coordinate of each version, and see that they are different. □

(3) $\big(\langle 2, -3, -7 \rangle \cdot \langle -1, 5, -2 \rangle \big) \langle -7, 3, -3 \rangle$

Solution. The expression $\big(\langle 2, -3, -7 \rangle \cdot \langle -1, 5, -2 \rangle \big)$ is a scalar, so the whole expression is a scalar multiple of $\langle -1, 5, -2 \rangle$ and is hence a vector. □

(It is $\langle 21, -9, 9 \rangle$.)

Problem 3 (10 points)**.** Let g be a real number. Find the distance from the point $(0, -1, g)$ to the plane given by $x + 2y - 3z = 2$.

Solution. Choose any point p on the plane, say $p = (0, 1, 0)$, and a normal vector to the plane, say $\mathbf{n} = \langle 1, 2, -3 \rangle$. Then find the length of the vector projection of $\langle 0, -1, g \rangle - p = \langle 0, -2, g \rangle$ onto $\langle 1, 2, -3 \rangle$ (equivalently, the absolute value of the scalar projection), which is

$$\frac{\big| \langle 0, -2, g \rangle \cdot \langle 1, 2, -3 \rangle \big|}{\| \langle 1, 2, -3 \rangle \|} = \frac{|(0)(1) + (-2)(2) + (g)(-3)|}{\sqrt{(1)^2 + (2)^2 + (-3)^2}} = \frac{|-4 - 3g|}{\sqrt{14}} = \frac{|4 + 3g|}{\sqrt{14}}.$$

□

One can use any other point on the plane in place of $(0, 1, 0)$. Probably the next easiest choice is $(2, 0, 0)$.

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Alternate solution. Use the formula in the book (top of page 870), to get

$$\frac{|(1)(0) + (2)(-1) + (-3)(g) - 2|}{\sqrt{(1)^2 + (2)^2 + (-3)^2}} = \frac{|-4 - 3g|}{\sqrt{14}} = \frac{|4 + 3g|}{\sqrt{14}}.$$

□

Problem 4 (8 points)**.** Let $\mathbf{u}$ and $\mathbf{v}$ be functions whose values are vectors in $\mathbb{R}^3$, and let $c(s) = \mathbf{u}(s) \cdot \mathbf{v}(s)$. Suppose that

$$\mathbf{u}(3) = \langle 2, 0, -4 \rangle, \qquad \mathbf{u}'(3) = \langle -5, 1, 3 \rangle, \qquad \mathbf{v}(3) = \langle 2, -1, 0 \rangle, \qquad \text{and} \qquad \mathbf{v}'(3) = \langle -3, 0, 1 \rangle.$$

Find $c'(3)$.

Solution. By the product rule for dot products, we have

$$c'(s) = \mathbf{u}'(s) \cdot \mathbf{v}(s) + \mathbf{u}(s) \cdot \mathbf{v}'(s).$$

Therefore

$$\begin{aligned} c'(3) &= \mathbf{u}'(3) \cdot \mathbf{v}(3) + \mathbf{u}(3) \cdot \mathbf{v}'(3) \\ &= \langle -5, 1, 3 \rangle \cdot \langle 2, -1, 0 \rangle + \langle 2, 0, -4 \rangle \cdot \langle -3, 0, 1 \rangle \\ &= \big[(-5)(2) + (1)(-1) + (3)(0)\big] + \big[(2)(-3) + (0)(0) + (-4)(1)\big] \\ &= -11 - 10 = -21. \end{aligned}$$

That is, $c'(3) = -21$. □

Problem 5 (6 points)**.** Find all real numbers s such that the vectors $\langle 2, -1, 3 \rangle$ and $\langle 3, 1-s, 1 \rangle$ are orthogonal.

Solution. Two vectors are orthogonal if and only if their dot product is zero. So we are looking for exactly the solutions to the equation

$$0 = \langle 2, -1, 3 \rangle \cdot 3, 1 - s, 1 \rangle = (2)(3) + (-1)(1 - s) + (3)(1) = 8 + s.$$

So the vectors $\langle 2, -1, 3 \rangle$ and $\langle 3, 1 - s, 1 \rangle$ are orthogonal if and only if $s = -8$. □

Problem 6 (10 points)**.** Find two *unit* vectors which are orthogonal to both $\langle -3, 4, -2 \rangle$ and $\langle 1, 2, -1 \rangle$.

Solution. Start by calculating a vector orthogonal to both the given ones:

$$\begin{aligned} \langle -3, 4, -2 \rangle \times \langle 1, 2, -1 \rangle &= \langle (4)(-1) - (-2)(2), (-2)(1) - (-3)(-1), (-3)(2) - (4)(1) \rangle \\ &= \langle 0, -5, -10 \rangle. \end{aligned}$$

We want two unit vectors parallel to this last expression, equivalently, parallel to $\langle 0, -5, -10 \rangle$. It has length $\sqrt{125} = 5\sqrt{5}$, so the answers are

$$\frac{1}{5\sqrt{5}} \langle 0, -5, -10 \rangle = \left\langle 0, -\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right\rangle \qquad \text{and} \qquad -\frac{1}{5\sqrt{5}} \langle 0, -5, -10 \rangle = \left\langle 0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle.$$

□

It is a bit easier if you observe that a vector is parallel to $\langle 0, -5, -10 \rangle$ if and only if it is parallel to $\langle 0, 1, 2 \rangle$.

Problem 7 (12 points)**.** Consider the surface in $\mathbb{R}^3$ given by

$$\frac{x^2}{4} + (y-2)^2 + \frac{z^2}{16} = 1.$$

Describe and draw its traces in the xz plane and the plane $x = \sqrt{3}$.

Solution. The trace in the xz plane is gotten by setting $y = 0$, which gives the equation

$$\frac{x^2}{4} + 4 + \frac{z^2}{16} = 1,$$

or

$$\frac{x^2}{4} + \frac{z^2}{16} = -3.$$

This equation has no solutions, so the trace is the empty set. No picture is included since there is nothing to draw, but a picture of just the x and z axes would also be appropriate, as long as you *say* that there is nothing to graph.

The trace in the plane $x = \sqrt{3}$ is given by the equation

$$\frac{3}{4} + (y-2)^2 + \frac{z^2}{16} = 1,$$

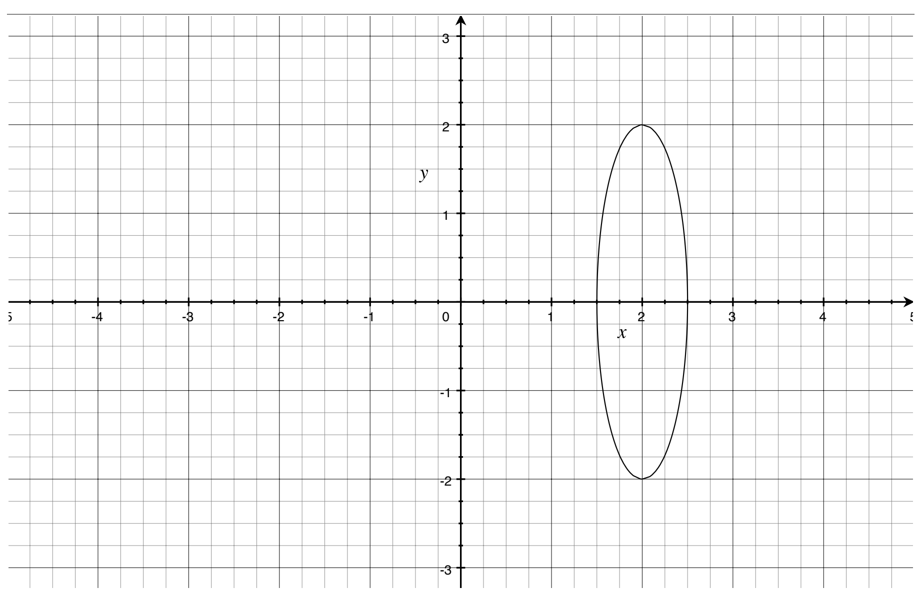
or

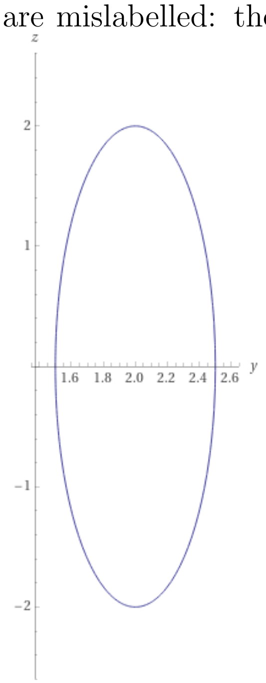
$$(y-2)^2 + \frac{z^2}{16} = \frac{1}{4}.$$

It is easiest to see what is happening if one rewrites it as

$$\frac{(y-2)^2}{\left(\frac{1}{2}\right)^2} + \frac{z^2}{2^2} = 1.$$

This is an ellipse with center at $(2, 0)$ (ordered pairs are (y, z)) and going through the points $(2 \pm \frac{1}{2},\ 0)$ and $(2,\ 0 \pm 2)$, that is, $(\frac{5}{2},\ 0)$, $(\frac{3}{2},\ 0)$, $(2, 2)$, and $(2, -2)$. Here are two versions of the graph, both made with low grade software. The axes in the left graph are mislabelled: they should by y (horizontal) and z (vertical).





As it says in the general instructions, axes must be labelled and have scales. ☐

Problem 8 (12 points)**.** Find an equation for the plane that contains the line $x(t) = 3t + 2$, $y(t) = -5t$, $z(t) = 4t + 5$ and contains the point $(1, 2, 3)$.

Solution. We find two directions in the plane, and take their cross product to get a normal vector.

The plane contains the points $(1, 2, 3)$ (given), $(2, 0, 5)$ (gotten by taking $t = 0$ in the given parametrization of the line), and $(5, -5, 9)$ (gotten by taking $t = 1$ in the given parametrization of the line). So take

$$\begin{aligned} \mathbf{n} &= [(2,0,5) - (1,2,3)] \times [(5,\ -5,\ 9) - (2,0,5)] \\ &= \langle 1,\ -2,\ 2 \rangle \times \langle 3,\ -5,\ 4 \rangle \\ &= \big\langle (-2)(4) - (2)(-5),\ (2)(3) - (1)(4),\ (1)(-5) - (-2)(3) \big\rangle \\ &= \langle 2,\ 2,\ 1 \rangle. \end{aligned}$$

Now the equation of the plane is

$$\langle 2,\ 2,\ 1 \rangle \cdot (\langle x, y, z \rangle - \langle 1, 2, 3 \rangle) = 0,$$

which is $2(x - 1) + 2(y - 2) + (z - 3) = 0$. This simplifies to $2x + 2y + z - 9 = 0$. (The simplification is *required*.) □

Alternate solution. Take the first direction in the plane to be $(2, 0, 5) - (1, 2, 3)$, as in the first solution, and take the second direction to be a direction vector for the given line, say $\langle 3,\ -5,\ 4 \rangle$. This gives the same cross product, and the same equation, as before.

There are many other possibilities. One simple one is to take the cross product in the other order, ending up with the equation $-2x - 2y - z + 9 = 0$. (It is easy to see that this equation determines the same plane.) □

Problem 9 (12 points)**.** Find parametric equations for the tangent line to the curve $\mathbf{r}(t) = \left\langle \frac{1}{3}t^3,\ t^2,\ 2t \right\rangle$ at the point $\left(\frac{1}{3}, 1, 2\right)$.

Solution. We have $\mathbf{r}(t) = \left(\frac{1}{3}, 1, 2\right)$ when $t = 1$. Start by calculating:

$$\mathbf{r}'(t) = \langle t^2,\ 2t,\ 2 \rangle.$$

So $\mathbf{r}'(1) = \langle 1, 2, 2 \rangle$. Take this to be the direction vector of the line. So the vector equation of the line is

$$\mathbf{l}(t) = \left\langle \tfrac{1}{3}, 1, 2 \right\rangle + t\langle 1, 2, 2 \rangle = \left\langle t + \tfrac{1}{3},\ 2t + 1,\ 2t + 2 \right\rangle.$$

We get the parametric equations $x(t) = t + \frac{1}{3}$, $y(t) = 2t + 1$, and $z(t) = 2t + 2$. □

Problem 10 (15 points)**.** Find the length of the curve $\mathbf{r}(t) = 4t\mathbf{i} + \frac{8}{3}t^{3/2}\mathbf{j} + t^2\mathbf{k}$ for $1 \le t \le 2$.

Solution. Start by calculating:

$$\mathbf{r}'(t) = 4\mathbf{i} + 4t^{1/2}\mathbf{j} + 2t\mathbf{k}.$$

So

$$\|\mathbf{r}'(t)\| = \sqrt{4^2 + (4t^{1/2})^2 + (2t)^2} = \sqrt{16 + 16t + 4t^2} = \sqrt{4(2 + t)^2} = 2(t + 2),$$

using $t + 2 \ge 0$ at the last step. Therefore the arc length is

$$\int_1^2 2(t + 2)\, dt = (t^2 + 4t)\Big|_1^2 = 16 - 5 = 11.$$

□

Remark. It is a little easier to write

$$\mathbf{r}'(t) = 2(2\mathbf{i} + 2t^{1/2}\mathbf{j} + t\mathbf{k}),$$

so

$$\begin{aligned}\|\mathbf{r}'(t)\| &= 2\|2\mathbf{i} + 2t^{1/2}\mathbf{j} + t\mathbf{k}\| = 2\sqrt{2^2 + (2t^{1/2})^2 + t^2} \\ &= 2\sqrt{4 + 4t + t^2} = 2\sqrt{(2+t)^2} = 2(t+2).\end{aligned}$$

(With more complicated expressions that factor out, this becomes more important.) □

Problem 11 (8 points)**.** Find the domain of the vector valued function

$$\mathbf{w}(t) = \left\langle \frac{1}{t+2}, \, -\sqrt{t+7}, \, \ln(8-t) \right\rangle.$$

Solution. The formula for the first coordinate imposes the restriction $t \neq -2$. The formula for the second coordinate imposes the restriction $t \geq -7$. The formula for the second coordinate imposes the restriction $t < 8$. So the domain consists of all point in $[-7, 8)$ except -2, that is, of the intervals $[-7, -2)$ and $(-2, 8)$. □

Remark. The correct mathematical notation is $[-7, -2) \cup (-2, 8)$. (Unfortunately, this is not in the book or in `WeBWorK`. The symbol "∪" is *not* the same as the capital letter "U", which is occasionally used in `WeBWorK` because there is no "∪" on the keyboard.) □

Problem 12 (17 extra credit points)**.** *Without* writing anything in terms of coordinates, prove the product rule for the derivative of the cross product of two vector valued functions. You may assume without proof that a differentiable vector valued function is continuous, and that the limit of a cross product is the cross product of the limits.

Solution. Let $\mathbf{u}$ and $\mathbf{v}$ be vector valued functions, and let $\mathbf{r}(t) = \mathbf{u}(t) \times \mathbf{v}(t)$. Assume $\mathbf{u}$ and $\mathbf{v}$ are differentiable at t. Then

$$\begin{aligned}\mathbf{r}'(t) &= \lim_{h \to 0} \frac{1}{h}[\mathbf{r}(t+h) - \mathbf{r}(t)] \\ &= \lim_{h \to 0} \frac{1}{h}\Big[\mathbf{u}(t+h) \times \mathbf{v}(t+h) - \mathbf{u}(t) \times \mathbf{v}(t)\Big] \\ &= \lim_{h \to 0} \frac{1}{h}\Big[\mathbf{u}(t+h) \times \mathbf{v}(t+h) - \mathbf{u}(t+h) \times \mathbf{v}(t) + \mathbf{u}(t+h) \times \mathbf{v}(t) - \mathbf{u}(t) \times \mathbf{v}(t)\Big] \\ &= \lim_{h \to 0} \frac{1}{h}\Big[\mathbf{u}(t+h) \times [\mathbf{v}(t+h) - \mathbf{v}(t)] + [\mathbf{u}(t+h) - \mathbf{u}(t)] \times \mathbf{v}(t)\Big] \\ &= \lim_{h \to 0} \left[\mathbf{u}(t+h) \times \frac{1}{h}[\mathbf{v}(t+h) - \mathbf{v}(t)]\right] + \lim_{h \to 0} \left[\frac{1}{h}[\mathbf{u}(t+h) - \mathbf{u}(t)] \times \mathbf{v}(t)\right] \\ &= \left(\lim_{h \to 0} \mathbf{u}(t+h)\right) \times \left(\lim_{h \to 0} \frac{1}{h}[\mathbf{v}(t+h) - \mathbf{v}(t)]\right) + \left(\lim_{h \to 0} \frac{1}{h}[\mathbf{u}(t+h) - \mathbf{u}(t)]\right) \times \mathbf{v}(t) \\ &= \mathbf{u}(t) \times \mathbf{v}'(t) + \mathbf{u}'(t) \times \mathbf{v}(t).\end{aligned}$$

This completes the proof. □

Problem 13 (18 extra credit points)**.** Reparametrize the curve $\mathbf{r}(t) = \langle 3\sin(\ln(t)), \, 4\ln(t), \, 3\cos(\ln(t)) \rangle$, for $t > 0$, with respect to arc length measured from the point where $t = 1$ in the direction of increasing t.

Solution. Start by calculating:

$$\mathbf{r}'(t) = \bigl\langle 3\cos(\ln(t))t^{-1},\, 4t^{-1},\, \sin(\ln(t))t^{-1} \bigr\rangle = t^{-1}\bigl\langle 3\cos(\ln(t)),\, 4,\, \sin(\ln(t)) \bigr\rangle.$$

So

$$\|\mathbf{r}'(t)\| = |t^{-1}|\sqrt{9\cos^2(\ln(t)) + 16 + 9\sin^2(\ln(t))} = t^{-1}\sqrt{25} = 5t^{-1},$$

using $t \geq 0$ at the second last step. Therefore the arc length function is

$$l(t) = \int_1^t 5u^{-1}\,du = 5\ln(t).$$

We will need the inverse function of l, which is $l^{-1}(s) = e^{s/5}$. Let's call the reparametrized curve $\mathbf{w}$. It is given by

$$\begin{aligned}\mathbf{w}(s) = \mathbf{r}(l^{-1}(s)) = \mathbf{r}(e^{s/5}) &= \bigl\langle 3\sin(\ln(e^{s/5})),\, 4\ln(e^{s/5}),\, 3\cos(\ln(e^{s/5})) \bigr\rangle \\ &= \bigl\langle 3\sin(s/5),\, 4s/5,\, 3\cos(s/5) \bigr\rangle.\end{aligned}$$

(The simplification is *required*.) We can rewrite it as $\mathbf{w}(t) = \bigl\langle 3\sin(t/5),\, 4t/5,\, 3\cos(t/5) \bigr\rangle$ if desired. □

Remark. A physicist would write this as follows:

$$\frac{d\mathbf{r}}{dt} = t^{-1}\bigl\langle 3\cos(\ln(t)),\, 4,\, \sin(\ln(t)) \bigr\rangle, \qquad \left\|\frac{d\mathbf{r}}{dt}\right\| = 5t^{-1}, \qquad s = \int_1^t \left\|\frac{d\mathbf{r}}{dt}\right\| dt = 5\ln(t),$$

so

$$t = e^{s/5},$$

and the reparametrization is

$$\mathbf{r} = \langle 3\sin(s/5),\, 4s/5,\, 3\cos(s/5) \rangle.$$

This is consistent with the view of $\mathbf{r}$ as the position of some particle, but obscures the fact that the original curve and the reparametrization are two rather different functions—as different as $y = e^{t/5}$ and $y = t$. □