MATH 281 (PHILLIPS): SOLUTIONS TO MIDTERM 1 SAMPLE PROBLEMS

Midterm 1 will be worth 100 points. Most problems will be similar to ones in `WeBWoprK`, written homework, examples in the book and in the lectures, and the sample problems below.

There are many more problems on this list than there will be on Midterm 1. The point values are intended to roughly indicate what fraction of the midterm each type of problem could be.

Problem 1 (2 points)**.** Is the expression $\langle 2, 3, -18\rangle \cdot \langle -1, 5, -2\rangle$ a scalar, a vector, or not defined?

Solution. A scalar. The dot product of two vectors is always a scalar. (In this case, it is 49.) ☐

Problem 2 (2 points)**.** If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathbb{R}^3$, is the expression $\mathbf{u} \times \mathbf{v} \times \mathbf{w}$ a scalar, a vector, or not defined?

Solution. Not defined, because it is ambiguous. (For example, $\mathbf{i} \times (\mathbf{j} \times \mathbf{j}) = 0$ but $(\mathbf{i} \times \mathbf{j}) \times \mathbf{j} = -\mathbf{i} \neq 0$. ☐

Problem 3 (2 points)**.** Is the expression $56 \times \langle 2, 3, -18\rangle$ a scalar, a vector, or not defined?

Solution. Not defined. The cross product of a scalar and a vector is not defined. ☐

Problem 4 (2 points)**.** Is the expression $\langle 2, -3\rangle \times \langle -1, 5\rangle$ a scalar, a vector, or not defined?

Solution. Not defined. The cross product of two vectors is defined only if they are both in $\mathbb{R}^3$. ☐

Problem 5 (2 points)**.** Is the expression

$$\langle 2, -3, -18\rangle \cdot (\langle -1, 5, -2\rangle \times \langle -7, 12, -27\rangle)$$

a scalar, a vector, or not defined?

Solution. A scalar. The expression $\langle -1, 5, -2\rangle \times \langle -7, 12, -27\rangle$ is a vector (namely $\langle -111,\ -13,\ 23\rangle$), and its dot product with $\langle 2, -3, -18\rangle$ is therefore a scalar (namely -597). ☐

Problem 6 (2 points)**.** Is the expression $56\mathbf{j} \times (2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})$ a scalar, a vector, or not defined?

Solution. A vector. It is the cross product of two vectors. (It happens to be $336\mathbf{i} - 112\mathbf{k}$.) ☐

Problem 7 (2 points)**.** Is the expression $26\mathbf{j} - (2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}) \cdot (2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})$ a scalar, a vector, or not defined?

Solution. Not defined. The expression $(2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}) \cdot (2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})$ is a scalar, and one can't subtract a scalar from a vector. ☐

Problem 8 (4 points)**.** Let $\mathbf{v} = \langle 2, 3, -6\rangle$ and $\mathbf{w} = \langle -1, 5, -2\rangle$. Find the cosine of the angle between $\mathbf{v}$ and $\mathbf{w}$.

Solution.

$$\cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} = \frac{-2 + 15 + 12}{\sqrt{4 + 9 + 36}\sqrt{1 + 25 + 4}} = \frac{25}{7\sqrt{30}}.$$

☐

Date: 22 October 2020.

Problem 9 (8 points)**.** Find parametric equations for the line through the point $(0,1,0)$, perpendicular to the line $x = 2t$, $y = 1 - t$, $z = 3t$, and parallel to the plane $x + y + z = 217$.

Solution. We need a direction vector $\mathbf{v}$ for the line. We get it by taking the cross product of a direction vector for the given line, say $\mathbf{u} = \langle 2, -1, 3\rangle$, and a normal vector to the given plane, say $\mathbf{n} = \langle 1, 1, 1\rangle$. (The resulting vector is parallel to the given plane and perpendicular to the given line.) So take $\mathbf{v} = \mathbf{u} \times \mathbf{n} = \langle -4, 1, 3\rangle$, giving a vector equation

$$\mathbf{r}(t) = \langle 0, 1, 0\rangle + t\langle -4, 1, 3\rangle = \langle -4t,\ 1 + t,\ 3t\rangle,$$

and parametric equations $x = -4t$, $y = 1 + t$, $z = 3t$. □

Other solutions are possible. Most obviously, one could use $\mathbf{n} \times \mathbf{u}$ in place of $\mathbf{u} \times \mathbf{n}$, getting the parametric equations $x = 4t$, $y = 1 - t$, $z = -3t$.

Problem 10 (7 points)**.** Find an equation for the plane that contains the line $x = 3t$, $y = 1-t$, $z = 2 - t$ and contains the point $(1, 2, 3)$.

Solution. We find two directions in the plane, and take their cross product to get a normal vector.

The plane contains the points $(1, 2, 3)$ (given), $(0, 1, 2)$ (gotten by taking $t = 0$ in the given parametrization of the line), and $(3, 0, 1)$ (gotten by taking $t = 1$ in the given parametrization of the line). So I will take

$$\mathbf{n} = [(1, 2, 3) - (0, 1, 2)] \times [(1, 2, 3) - (3, 0, 1)] = \langle 0, -4, 4\rangle.$$

Now the equation of the plane is

$$\langle 0, -4, 4\rangle \cdot (\langle x, y, z\rangle - \langle 1, 2, 3\rangle) = 0,$$

which is $-4(y - 2) + 4(z - 3) = 0$. This simplifies to $-4y + 4z - 4 = 0$. (The simplification is *required*.) □

Alternate solution. Take the first direction in the plane to be $(1, 2, 3) - (0, 1, 2)$, as in the first solution, and take the second direction to be a direction vector for the given line, say $\langle 3, -1, -1\rangle$. This gives the same cross product, and the same equation, as before. □

There are many other possibilities. One simple one is that I could have taken the cross product in the other order, ending up with the equation $4y - 4z + 4 = 0$. (It is easy to see that this equation determines the same plane.)

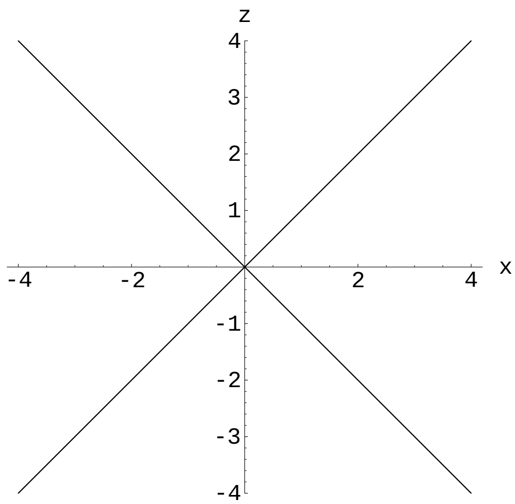
Problem 11 (8 points)**.** Find the traces of the surface $z^2 = x^2 + y^2$ in the xz and yz planes, and in the planes $z = k$ for general k. Then identify the surface and sketch it.

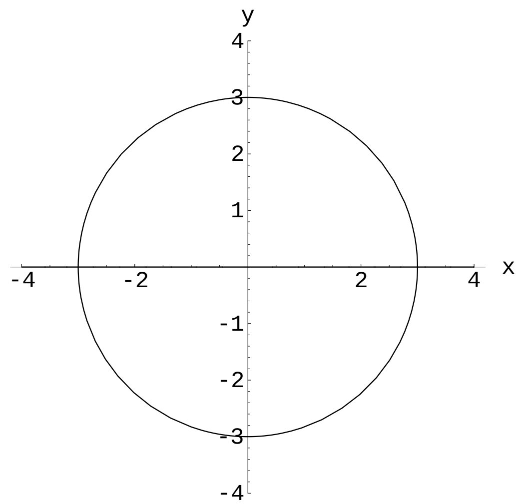
Solution. The trace in the xz plane is $z^2 = x^2$ (set $y = 0$), which is $z = \pm x$, that is, two lines, of slopes ± 1, which cross at the origin.

The trace in the yz plane is the same, by the same calculation.

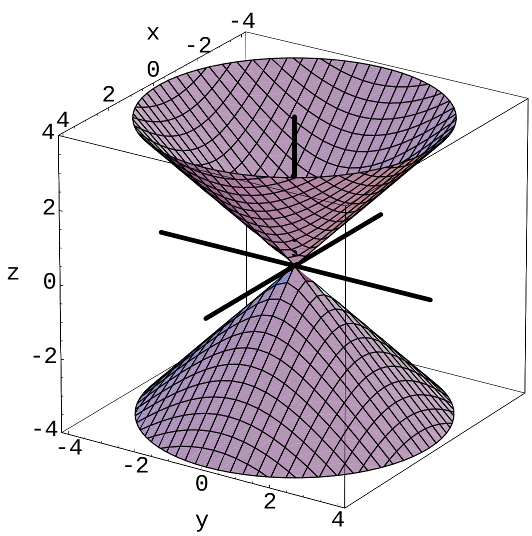
The trace in the plane $z = k$ is $x^2 + y^2 = k^2$, which is a circle with center $(0, 0)$ and radius $|k|$. (Note: *not* radius k.)

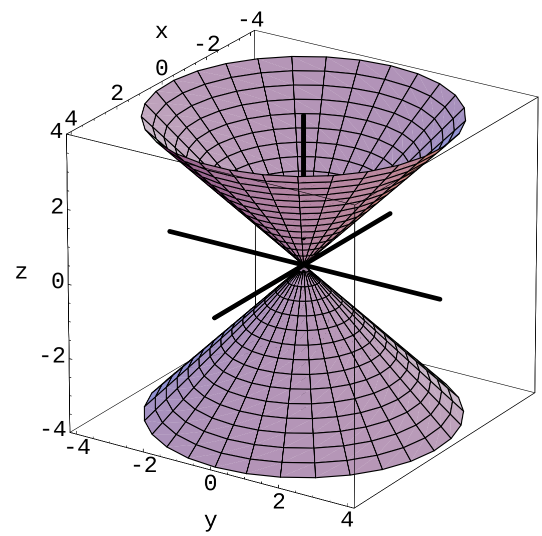
Here are traces in the xz plane (left) and the plane $z = 3$ (right):





Therefore the surface is a cone aligned along the z-axis, with vertex at the origin (pointing both up and down). Here are two plots, differing in the way the mesh is drawn on the surface:





$\square$

Problem 12 (14 points)**.** Let $\mathbf{r}(t) = \langle t^{-1},\, t^2,\, -t^{-1}\ln(t)\rangle$. Find parametric equations for the tangent line to this curve at the point $(1, 1, 0)$.

Solution. We have $\mathbf{r}(t) = (1, 1, 0)$ for $t = 1$. So $\mathbf{r}'(1)$ is a direction vector for the line. Now

$$\mathbf{r}'(t) = \langle -t^{-2},\, 2t,\, t^{-2}\ln(t) - t^{-1}(1/t)\rangle = \langle -t^{-2},\, 2t,\, t^{-2}(\ln(t) - 1)\rangle,$$

so $\mathbf{r}'(1) = \langle -1,\, 2,\, -1\rangle$. The line through $(1, 1, 0)$ and with this direction has the vector equation

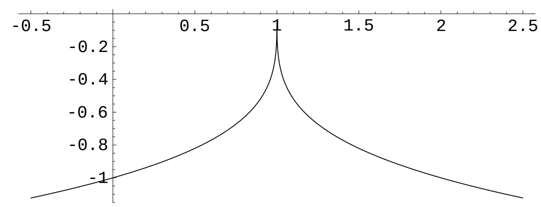
$$\mathbf{l}(t) = \langle 1, 1, 0\rangle + t\langle -1,\, 2,\, -1\rangle = \langle 1 - t,\, 1 + 2t,\, -t\rangle,$$

and parametric equations $x = 1 - t$, $y = 1 + 2t$, $z = -t$. $\square$

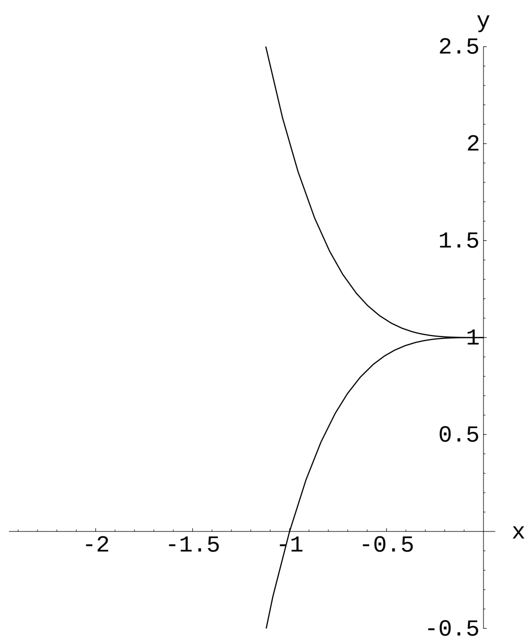
Warning: If you use $\mathbf{r}'(t)$ instead of $\mathbf{r}'(1)$ in the equation for the line, you will get something that is not the equation of any line.

Problem 13 (7 points)**.** Sketch the curve with the vector equation $\mathbf{r}(t) = \langle -t^2, 1+t^7 \rangle$. Indicate with an arrow the direction in which t increases.

Solution. The curve lies on the graph of the equation $-x^7 = (y-1)^2$. (Check that every point $\langle -t^2, 1+t^7 \rangle$ has x and y coordinates which satisfy this equation!) We can rewrite the equation as $x = -(y-1)^{2/7}$. The function $f(x) = -(x-1)^{2/7}$ is of a type that you have probably seen before; here is a graph:



To get the curve, we reflect this graph in the line $y = x$, getting:



The direction in which t increases is upwards. (The graph does not have the arrow showing the direction.) $\square$

Problem 14 (12 points)**.** Find the length of the curve

$$\mathbf{r}(t) = \langle t^2,\, \sin(t) - t\cos(t),\, \cos(t) + t\sin(t) \rangle$$

for $0 \leq t \leq \pi$.

Solution. First differentiate: $\mathbf{r}'(t) = \langle 2t,\, t\sin(t),\, t\cos(t) \rangle$ (using the product rule and cancelling appropriately in the last two terms). Therefore

$$|\mathbf{r}'(t)| = \sqrt{4t^2 + t^2\sin^2(t) + t^2\cos^2(t)} = \sqrt{t^2(4 + \sin^2(t) + \cos^2(t))} = \sqrt{5t^2}.$$

Since $t \geq 0$ in the range of interest, we get $|\mathbf{r}'(t)| = \sqrt{5} \cdot t$. So the length is

$$\int_0^\pi \sqrt{5} \cdot t \, dt = \frac{\sqrt{5} \cdot \pi^2}{2}.$$

□

Problem 15 (4 points)**.** Find an equation of the sphere that passes through the origin and whose center is $(1, -2, 3)$.

Solution. The square of the radius is

$$\|\langle 1,\ -2,\ 3\rangle\|^2 = (1)^2 + (-2)^2 + (3)^2 = 14,$$

so the equation is $(x-1)^2 + (y+2)^2 + (z-3)^2 = 14$. (This is as simple a form as can be found, so it need not be multiplied out.) □

Problem 16 (6 points)**.** Find two *unit* vectors which are orthogonal to both $\langle 1, 1, 1\rangle$ and $\langle 2, 0, 1\rangle$.

Solution. Start by calculating a vector orthogonal to both the given ones:

$$\langle 1, 1, 1\rangle \times \langle 2, 0, 1\rangle = \langle 1, 1, -2\rangle.$$

We want two unit vectors parallel to this last expression, equivalently, parallel to $\langle 1, 1, -2\rangle$. It has length $\sqrt{6}$, so the answers are

$$\frac{1}{\sqrt{6}}\langle 1, 1, -2\rangle \quad \text{and} \quad -\frac{1}{\sqrt{6}}\langle 1, 1, -2\rangle.$$

□

Problem 17 (3 points)**.** Let s and t be real numbers. Let $\mathbf{a} = s\mathbf{i} - 2s\mathbf{j} + 3s\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - t\mathbf{j} - 5t\mathbf{k}$. Find $\mathbf{a} \cdot \mathbf{b}$.

Solution.

$$\mathbf{a} \cdot \mathbf{b} = (s)(1) + (-2s)(-t) + (3s)(-5t) = s - 13st.$$

□

Problem 18 (6 points)**.** Let t be a real number. Let $\mathbf{a} = \mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - e^t\mathbf{j} - e^{-t}\mathbf{k}$. Find $\mathbf{a} \times \mathbf{b}$.

Solution.

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \langle (e^t)(-e^{-t}) - (e^{-t})(e^t),\ -[(1)(e^{-t}) - (e^{-t})(2)],\ (1)(-e^t) - (e^t)(2)\rangle \\ &= \langle 0,\ 3e^{-t},\ -3e^t\rangle. \end{aligned}$$

□

Problem 19 (6 points)**.** Find the vector projection of $\langle 12, 1, 2\rangle$ onto $\langle -1, 4, 8\rangle$.

Solution. Using the formula in the book, we get

$$\begin{aligned} \left(\frac{\langle 12, 1, 2\rangle \cdot \langle -1, 4, 8\rangle}{\|\langle -1, 4, 8\rangle\|^2}\right)\langle -1, 4, 8\rangle &= \left(\frac{(12)(-1) + (1)(4) + (2)(8)}{(-1)^2 + (4)^2 + (8)^2}\right)\langle -1, 4, 8\rangle \\ &= \left(\frac{8}{81}\right)\langle -1, 4, 8\rangle = \left\langle -\frac{8}{81}, \frac{32}{81}, \frac{64}{81}\right\rangle. \end{aligned}$$

□

Problem 20 (8 points)**.** Find the distance from the point $(1, -2, 4)$ to the plane given by $3x + 2y + 6z = 5$.

Solution. Choose any point p on the plane, say $p = (1, 1, 0)$, and a normal vector to the plane, say $\mathbf{n} = \langle 3, 2, 6 \rangle$. Then find the length of the vector projection of $\langle 1,\, -2,\, 4 \rangle - p = \langle 0,\, -3,\, 4 \rangle$ onto $\langle 3, 2, 6 \rangle$ (equivalently, the absolute value of the scalar projection), which is

$$\frac{\bigl|\langle 0,\, -3,\, 4 \rangle \cdot \langle 3, 2, 6 \rangle\bigr|}{\|\langle 3, 2, 6 \rangle\|} = \frac{|(0)(3) + (-3)(2) + (4)(6)|}{\sqrt{(3)^2 + (2)^2 + (6)^2}} = \frac{18}{\sqrt{49}} = \frac{18}{7}.$$

□

Problem 21 (6 points)**.** Find the angle between the planes given by $9x - 3y + 6z = 2$ and $2y = 6x + 4z$.

Solution. The equations can be rewritten as

$$3x - y + 2z = \frac{2}{3} \qquad \text{and} \qquad 3x - y + 2z = 0.$$

Therefore the planes don't intersect, and the angle between them is not defined. (They are parallel.) □

You can see this happening if you try the standard method. Let $\mathbf{m}$ and $\mathbf{n}$ be normal vectors to the planes. Then we want the acute angle θ between the directions of $\mathbf{m}$ and $\mathbf{n}$. It is determined by

$$\cos(\theta) = \frac{|\mathbf{m} \cdot \mathbf{n}|}{\|\mathbf{m}\| \|\mathbf{n}\|}.$$

Rewrite the second equation as $-6x + 2y - 4z = 0$. Then the obvious choices for m and n are $\langle 9,\, -3,\, 6 \rangle$ and $\langle -6,\, 2,\, -4 \rangle$. The arithmetic is easier with the choices $\mathbf{m} = \langle 3,\, -1,\, 2 \rangle$ and $\mathbf{n} = \langle -3,\, 1,\, -2 \rangle$, which are nonzero scalar multiples of the vectors above. With these choices,

$$\begin{aligned}\cos(\theta) = \frac{|\mathbf{m} \cdot \mathbf{n}|}{\|\mathbf{m}\| \|\mathbf{n}\|} &= \frac{\bigl|\langle 3,\, -1,\, 2 \rangle \cdot \langle -3,\, 1,\, -2 \rangle\bigr|}{\|\langle 3,\, -1,\, 2 \rangle\| \|\langle -3,\, 1,\, -2 \rangle\|} \\ &= \frac{\bigl|(3)(-3) + (-1)(1) + (2)(-2)\bigr|}{\sqrt{(3)^2 + (-1)^2 + (2)^2}\sqrt{(-3)^2 + (1)^2 + (-2)^2}} = \frac{16}{\sqrt{16}\sqrt{16}} = 1.\end{aligned}$$

Therefore $\theta = \arccos(1) = 0$.

This outcome tells you that either the planes are identical, or they are parallel and don't intersect. Since $(0, 0, 0)$ is on the second one but not the first, hey are parallel and don't intersect. So there is no angle between them.

Problem 22 (8 points)**.** Find all real numbers t such that the vectors $\langle 1,\, 2 - t,\, -1 \rangle$ and $\langle -3,\, -2,\, 1 - 2t \rangle$ are orthogonal.

Solution. Two vectors are orthogonal if and only if their dot product is zero. So we are looking for exactly the solutions to the equation

$$0 = \langle 1,\, 2 - t,\, -1 \rangle \cdot \langle -3,\, -2,\, 1 - 2t \rangle = (1)(-3) + (2 - t)(-2) + (-1)(1 - 2t) = 4t - 8.$$

So the vectors $\langle 1,\, 2 - t,\, -1 \rangle$ and $\langle -3,\, -2,\, 1 - 2t \rangle$ are orthogonal if and only if $t = 2$. □

Problem 23 (8 points)**.** Find all real numbers s such that the vectors $\langle -12,\, s,\, 4 \rangle$ and $\langle -3,\, -2,\, -1 \rangle$ are parallel.

Solution. Two vectors in $\mathbb{R}^3$ are parallel if and only if their cross product is zero. So we are looking for exactly the solutions to the equation

$$\begin{aligned} 0 &= \langle -12,\, s,\, 4\rangle \times \langle -3,\, -2,\, -1\rangle \\ &= \big\langle (s)(-1) - (4)(-2),\, (4)(-3) - (-12)(-1),\, (-12)(-2) - (s)(-3)\big\rangle \\ &= \langle 8 - s,\, -24,\, 3s + 24\rangle. \end{aligned}$$

Since $-24 \neq 0$, there are no solutions. Therefore there is no real number s such that the vectors $\langle -12,\, s,\, 4\rangle$ and $\langle -3,\, -2,\, -1\rangle$ are parallel. □

Alternate solution. Since both vectors are nonzero, they are parallel if and only if there is a real number λ such that $\langle -12,\, s,\, 4\rangle = \lambda\langle -3,\, -2,\, -1\rangle$. This gives rise to the equations

$$-12 = -3\lambda, \qquad s = -2\lambda, \qquad \text{and} \qquad 4 = -\lambda.$$

The first equation says $\lambda = 4$, while the last says $\lambda = -4$, so there is no such λ. □

Problem 24 (8 points)**.** Find all real numbers c such that

$$\det\begin{pmatrix} -1 & c & 4 \\ 3 & -2 & -1 \\ -3 & 1 & 2 \end{pmatrix} = 0.$$

Solution. We have

$$\begin{aligned} &\det\begin{pmatrix} -1 & c & 4 \\ 3 & -2 & -1 \\ -3 & 1 & 2 \end{pmatrix} \\ &\quad = (-1)[(-2)(2) - (-1)(1)] - c[(3)(2) - (-1)(-3)] + (4)[(3)(1) - (-2)(-3)] \\ &\quad = -9 - 3c. \end{aligned}$$

This is zero if and only if $c = -3$. □

Problem 25 (6 points)**.** Let t be a real number. Find the volume of the parallelepiped determined by the vectors

$$\mathbf{a} = \mathbf{i} + \mathbf{j} - \mathbf{k}, \quad \mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}, \quad \text{and} \quad \mathbf{c} = -\mathbf{i} + \mathbf{j} + t\mathbf{k}.$$

Solution. It is

$$|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = |(\mathbf{i} + \mathbf{j} - \mathbf{k}) \cdot (-2\mathbf{i} - (1 + t)\mathbf{j})| = |-2 - (1 + t)| = |3 + t|.$$

(Note that $3 + t$ is *not* correct, since $3 + t$ might be negative.) □

Problem 26 (7 points)**.** Give an example of three distinct lines L_1, L_2, and L_3 in $\mathbb{R}^3$ such that L_2 and L_3 are both perpendicular to L_1, but L_2 and L_3 are not parallel.

Solution. Take L_1 to be the x-axis, L_2 to be the y-axis, and L_3 to be the z-axis. □

Many other solutions are possible.

Problem 27 (7 points)**.** Find a vector equation and parametric equations for the line through the point $(-2,\, 0,\, 1)$ and parallel to the line $x = 2t$, $y = 1 - t$, $z = 4 + 3t$.

Solution. We can take the direction vector to be $\mathbf{v} = \langle 2,\, -1,\, 3\rangle$ (using the coefficients of t in the parametric equation for the given line). Then the vector equation is

$$\mathbf{r}(t) = \langle -2,\, 0,\, 1\rangle + t\langle 2,\, -1,\, 3\rangle = \langle -2 + 2t,\, -t,\, 1 + 3t\rangle,$$

and the parametric equations are $x = -2 + 2t$, $y = -t$, $z = 1 + 3t$. □

Problem 28 (7 points)**.** Find a vector equation and parametric equations for the line through the points $(6,\ 1,\ -3)$ and $(2,4,5)$.

Solution. We can take the direction vector to be

$$\mathbf{v} = \langle 2,4,5\rangle - \langle 6,\ 1,\ -3\rangle = \langle -4,\ 3,\ 8\rangle.$$

Then the vector equation is

$$\mathbf{r}(t) = \langle 6,\ 1,\ -3\rangle + t\langle -4,\ 3,\ 8\rangle = \langle 6-4t,\ 1+3t,\ -3+8t\rangle,$$

and the parametric equations are $x = 6-4t$, $y = 1+3t$, $z = -3+8t$. □

Alternate solution. Other, equivalent, answers will be obtained if one instead uses

$$\mathbf{v} = \langle 6,\ 1,\ -3\rangle - \langle 2,4,5\rangle = \langle 4,\ -3,\ -8\rangle,$$

or if one uses the other point for the constant term. □

Problem 29 (7 points)**.** Find a vector equation and parametric equations for the line through the point $(-2,\ 1,\ 0)$ and perpendicular to both $\mathbf{i}+\mathbf{j}$ and $\mathbf{j}+\mathbf{k}$.

Solution. We need a direction vector $\mathbf{v}$ which is orthogonal to both $\mathbf{i}+\mathbf{j}$ and $\mathbf{j}+\mathbf{k}$. We can get one by forming the cross product:

$$\mathbf{v} = (\mathbf{i}+\mathbf{j}) \times (\mathbf{j}+\mathbf{k}) = \langle 1,\ -1,\ 1\rangle.$$

Then the vector equation is

$$\mathbf{r}(t) = \langle -2,\ 1,\ 0\rangle + t\langle 1,\ -1,\ 1\rangle = \langle -2+t,\ 1-t,\ t\rangle,$$

and the parametric equations are $x = -2+t$, $y = 1-t$, $z = t$. □

Problem 30 (7 points)**.** Find an equation for the plane that contains the line $x = 3+2t$, $y = t$, $z = 8-t$ and is parallel to the plane $x+2y+4z = 17$.

Solution. For there to be any solution at all, the given line had better be parallel to the given plane. It is convenient to check this after finding a normal vector to the plane.

It is easy to find a normal vector: use the coefficients of the equation for the given plane, so take $\mathbf{n} = \langle 1,\ 2,\ 4\rangle$.

The given line has $\mathbf{v} = \langle 2,\ 1,\ -1\rangle$ as a direction vector. We have $\mathbf{v}\cdot\mathbf{n} = 0$, so the line is orthogonal to the normal vector to the plane, hence parallel to the plane. Therefore a solution exists, and can be found by taking any plane parallel to the given one and going through any point on the given line.

We need a point on the plane, and I got $(3,0,8)$ by taking $t = 0$ in the parametrization of the line. The equation of the plane is now $\langle 1,\ 2,\ 4\rangle \cdot (\langle x,y,z\rangle - \langle 3,0,8\rangle) = 0$, which is $(x-3)+2y+4(z-8) = 0$. This simplifies to $x+2y+4z-35 = 0$. (The simplification is *required*.) □

Problem 31 (8 points)**.** Find an equation for the plane through the points $(0,1,2)$, $(2,\ -3,\ 8)$, and $(5,2,5)$.

Solution. We find two directions in the plane, and take their cross product to get a normal vector. I will take

$$\mathbf{n} = [(2,\ -3,\ 8) - (0,1,2)] \times [(5,2,5) - (0,1,2)] = \langle -18,\ 24,\ 22\rangle.$$

Now the equation of the plane is

$$\langle -18,\ 24,\ 22\rangle \cdot (\langle x,y,z\rangle - \langle 0,1,2\rangle) = 0,$$

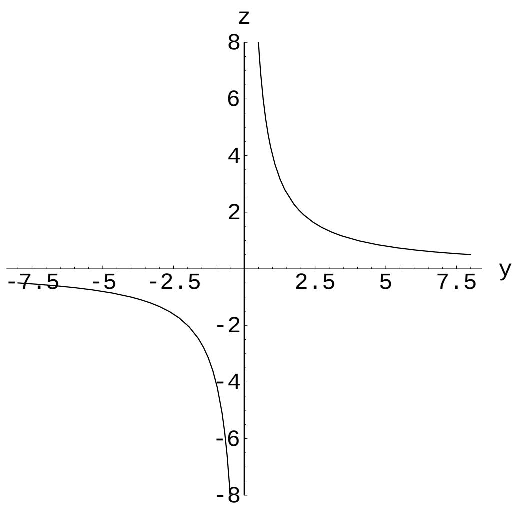
which is $-18x + 24(y-1) + 22(z-2) = 0$. This simplifies to $-18x + 24y + 22z - 68 = 0$. (The simplification is *required*.) $\square$

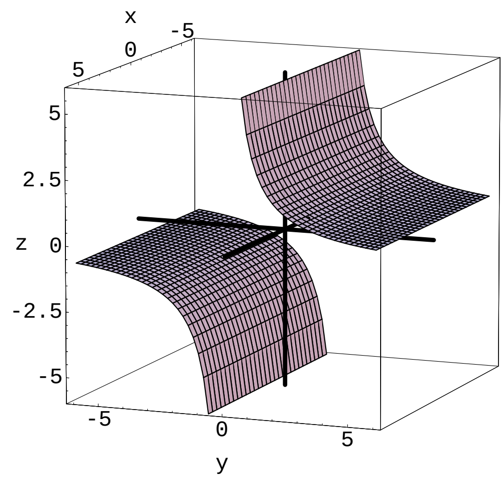
Alternate solution. One could take $\mathbf{n}$ to be the cross product of any two differences of the three points, as long as all three points are used. Also, in the expression $\langle x, y, z\rangle - \langle 0, 1, 2\rangle$, one could use either of the other two points in place of $(0, 1, 2)$. $\square$

Problem 32 (10 points)**.** Describe and sketch the surface $yz = 4$. Include a description of appropriate traces.

Solution. Since x does not appear, it is a cylinder oriented along the x-axis. The trace in every plane parallel to the yz plane is the same hyperbola, namely $yz = 4$.

Here are the trace in the yz plane (left) and the surface (right):





$\square$

Remark. I would have liked to have made the grid lines in this graph equally spaced. Doing so with the program I used requires reparametrizing the curve $\mathbf{r}(t) = (t, 4/t)$ by arc length.

Problem 33 (12 points)**.** Let $\mathbf{r}(t) = \arctan(t)\mathbf{i} + e^{-2t}\mathbf{j} - (\ln(t)/t)\mathbf{k}$. Find $\lim_{t\to\infty} \mathbf{r}(t)$.

Solution. The limit $\lim_{t\to\infty}(-\ln(t)/t)$ has the indterminate form "$-\frac{\infty}{\infty}$", so we try L'Hopital's Rule:

$$\lim_{t\to\infty}\left(-\frac{\ln(t)}{t}\right) = -\lim_{t\to\infty}\frac{\left(\frac{1}{t}\right)}{1} = 0.$$

Therefore

$$\lim_{t\to\infty}\arctan(t) = \tfrac{\pi}{2}, \qquad \lim_{t\to\infty} e^{-2t} = 0, \qquad \text{and} \qquad \lim_{t\to\infty}\left(-\frac{\ln(t)}{t}\right) = 0,$$

so $\lim_{t\to\infty}\mathbf{r}(t) = \left\langle \frac{\pi}{2}, 0, 0\right\rangle$. $\square$

Problem 34 (12 points)**.** Let $\mathbf{r}(t) = \arccos(t)\mathbf{i} + e^{-2t}\mathbf{j} - \sin(t)e^t\mathbf{k}$. Find $\mathbf{r}'(t)$.

Solution. Use the chain rule on the second term and the product rule on the third term to get

$$\begin{aligned}\mathbf{r}'(t) &= \left(-\frac{1}{\sqrt{1-x^2}}\right)\mathbf{i} + (-2)e^{-2t}\mathbf{j} - [\cos(t)e^t + \sin(t)e^t]\mathbf{k}\\ &= \left(-\frac{1}{\sqrt{1-x^2}}\right)\mathbf{i} - 2e^{-2t}\mathbf{j} - [\cos(t) + \sin(t)]e^t\mathbf{k}.\end{aligned}$$

□

Problem 35 (15 points)**.** Let $\mathbf{r}(t) = \langle t^{-1}, \sin(\pi t), t^2/(1+t^6)\rangle$. Find $\int_1^2 \mathbf{r}(t)\,dt$.

Solution. Integrate coordinatewise:

$$\int_1^2 t^{-1}(t)\,dt = \ln(2) - \ln(1) = \ln(2),$$

$$\int_1^2 \sin(\pi t)\,dt = \frac{1}{\pi}[-\cos(2\pi) - (-\cos(\pi))] = -\frac{2}{\pi},$$

and

$$\int_1^2 \frac{t^2}{1+t^6}\,dt = \frac{1}{3}\arctan(t^3)\Big|_1^2 = \frac{1}{3}[\arctan(8) - \arctan(1)] = \frac{\arctan(8)}{3} - \frac{\pi}{12}.$$

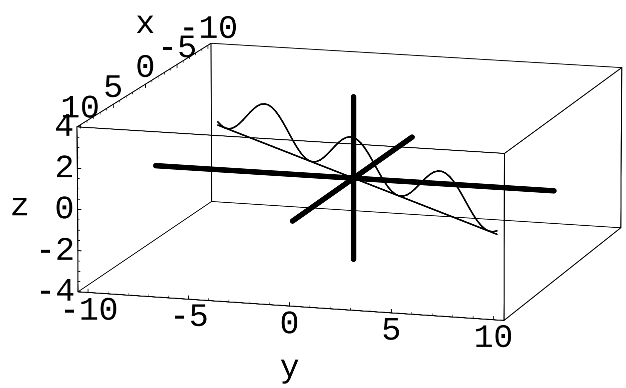
So

$$\int_1^2 \mathbf{r}(t)\,dt = \left\langle \ln(2),\ -\frac{2}{\pi},\ \frac{\arctan(8)}{3} - \frac{\pi}{12} \right\rangle.$$

□

Problem 36 (10 points)**.** Sketch the curve with the vector equation $\mathbf{r}(t) = \langle t, t, 1+\cos(t)\rangle$. Indicate with an arrow the direction in which t increases.

Solution. It looks like a graph over the line $y = x$ in the xy plane. Here is a picture (without the requested arrow), showing both the curve and the line $y = x$ in the xy plane.



In the picture, t increases from left to right. □

Problem 37 (10 points)**.** Let $\mathbf{u}$ and $\mathbf{v}$ be functions whose values are vectors in $\mathbb{R}^3$, and let $f(t) = \mathbf{u}(t)\cdot\mathbf{v}(t)$. Suppose that

$\mathbf{u}(4) = \langle 2, 0, 4\rangle$, $\mathbf{u}'(4) = \langle 0, 1, 3\rangle$, $\mathbf{v}(4) = \langle 2, -1, 0\rangle$, and $\mathbf{v}'(4) = \langle -3, 0, 1\rangle$.

Find $f'(4)$.

Solution. By the product rule for dot products, we have

$$f'(t) = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t).$$

Therefore

$$\begin{aligned} f'(4) &= \mathbf{u}'(4)\cdot\mathbf{v}(4) + \mathbf{u}(4)\cdot\mathbf{v}'(4) \\ &= \langle 0,1,3\rangle\cdot\langle 2,-1,0\rangle + \langle 2,0,4\rangle\cdot\langle -3,0,1\rangle = -1-2 = -3. \end{aligned}$$

□

Problem 38 (8 points)**.** Let $\mathbf{w}$ and $\mathbf{w}$ be functions whose values are vectors in $\mathbb{R}^3$, and let $\mathbf{r}(t) = \mathbf{v}(t) \times \mathbf{w}(t)$. Suppose that

$$\mathbf{v}(3) = \langle 2, 0, 5\rangle, \quad \mathbf{v}'(3) = \langle 0, 1, -3\rangle, \quad \mathbf{w}(3) = \langle -2, -1, 0\rangle, \quad \text{and} \quad \mathbf{w}'(3) = \langle -3, 0, -1\rangle.$$

Find $\mathbf{r}'(3)$.

Solution. By the product rule for cross products, we have

$$\mathbf{r}'(t) = \mathbf{v}'(t) \times \mathbf{w}(t) + \mathbf{v}(t) \times \mathbf{w}'(t).$$

Therefore

$$\begin{aligned} f'(3) &= \mathbf{v}'(3) \times \mathbf{w}(3) + \mathbf{v}(3) \times \mathbf{w}'(3) \\ &= \langle 2, 0, 5\rangle \times \langle -2, -1, 0\rangle + \langle 2, 0, 5\rangle \times \langle -3, 0, -1\rangle = -1 - 2 = -3. \end{aligned}$$

□

Problem 39 (12 points)**.** Let $\mathbf{r}(t) = \langle \sin(t), \cos(t), t^{3/2}\rangle$. Find the arc length of this curve over the interval $2 \le t \le 4$.

Solution. We have

$$\mathbf{r}'(t) = \left\langle \cos(t), -\sin(t), \left(\frac{3}{2}\right) t^{1/2} \right\rangle.$$

Therefore

$$\|\mathbf{r}'(t)\| = \sqrt{(\cos(t))^2 + (-\sin(t))^2 + \left(\left(\frac{3}{2}\right) t^{1/2}\right)^2} = \sqrt{1 + \frac{9t}{4}}$$

Using the substitution $u = 1 + \frac{9t}{4}$, so $dt = \frac{4}{9}\, du$, we get

$$\int \sqrt{1 + \frac{9t}{4}}\, dt = \frac{4}{9} \int u^1 2\, du = \left(\frac{4}{9}\right)\left(\frac{2}{3}\right) u^{3/2} + C = \left(\frac{8}{27}\right) u^{3/2} + C.$$

The arc length is then

$$\int_2^4 \|\mathbf{r}'(t)\| = \int_2^4 \sqrt{1 + \frac{9t}{4}}\, dt = \left(\frac{8}{27}\right) u^{3/2}\Big|_2^4 = \left(\frac{8}{27}\right) 4^{3/2} - \left(\frac{8}{27}\right) 2^{3/2} = \frac{64}{27} - \frac{16\sqrt{2}}{27}.$$

□

Problem 40 (12 points)**.** Let $\mathbf{r}(t) = \arctan(2t)\mathbf{i} + \cos(t^2 + t)\mathbf{j} + t^{3/2}\mathbf{k}$. Set up, but do not attempt to evaluate, an integral which gives the arc length of this curve over the interval $3 \le t \le 5$.

Solution. We have

$$\begin{aligned} \mathbf{r}'(t) &= \left\langle \frac{2}{1 + (2t)^2}, -\sin(t^2 + t)(2t + 1), \left(\frac{3}{2}\right) t^{1/2} \right\rangle \\ &= \left\langle \frac{2}{1 + 4t^2}, -(2t + 1)\sin(t^2 + t), \left(\frac{3}{2}\right) t^{1/2} \right\rangle. \end{aligned}$$

Therefore

$$\begin{aligned}\|\mathbf{r}'(t)\| &= \sqrt{\left(\frac{2}{1+4t^2}\right)^2 + (-(2t+1)\sin(t^2+t))^2 + \left(\left(\frac{3}{2}\right)t^{1/2}\right)^2} \\ &= \sqrt{\frac{4}{(1+4t^2)^2} + (2t+1)^2\sin^2(t^2+t) + \frac{9t}{4}}.\end{aligned}$$

The arc length is then

$$\int_3^5 \|\mathbf{r}'(t)\| = \int_3^5 \sqrt{\frac{4}{(1+4t^2)^2} + (2t+1)^2\sin^2(t^2+t) + \left(\frac{9t}{4}\right)}\, dt.$$

□