SOLUTIONS TO MIDTERM 2 SAMPLE PROBLEMS

Midterm 2 will cover through the end of Section 14.6 of the book. Emphasis will be on material covered since the first midterm, but it is cumulative, and material from before the first midterm will appear on Midterm 2.

Most problems will be similar to `WeBWorK` problems, written homework problems, problems from the real and sample Midterm 1, and the problems here. Note, though, that the exact form of functions which appear could vary substantially. In particular, a problem requiring computation of the limit or derivative of a single variable function could be modified to use any other single variable function; the methods required to find the limit or derivative could be very different. (This includes computations of partial derivatives, which are really just derivatives of single variable functions.)

Be sure to get the notation right! (This is a frequent source of errors.) The right notation will help you get the mathematics right, and I will complain about incorrect notation.

This handout contains two parts. First is an old exam, from a slightly different course, and which doesn't include anything from Midterm 1. Second is a collection of extra problems. The problems have point values attached, which give a rough idea of the point values problems requiring a similar amount of work will have on the real exam. Note, however, that the real midterm will total 100 points, and the problems here total much more than that.

An old Midterm 2, from a sightly different course and without any review problems.

Problem 1 (10 points)**.** Describe the domain of the function $f(x,y) = \sqrt{x^2+y^2-1} + \ln(4 - x^2 - y^2)$. Give reasons.

Solution. The domain consists of all (x, y) in $\mathbb{R}^2$ such that $1 \leq \sqrt{x^2+y^2} < 2$, equivalently, $1 \leq x^2 + y^2 < 4$. This is the region between the circles of radius 1 and radius 2 centered at the origin, including the inner one but not the outer one.

The reason is that $\sqrt{x^2+y^2-1}$ is defined if and only if $x^2 + y^2 \geq 1$, while $\ln(4 - x^2 - y^2)$ is defined if and only if $x^2 + y^2 < 4$. The point (x, y) is in the domain of f if and only if both expressions are defined.

It is not correct to write "~~x, y in $\mathbb{R}^2$~~". This answer will receive very little credit, because it tries to describe a subset of $\mathbb{R}$ rather than a subset of $\mathbb{R}^2$. □

Problem 2 (10 points/part)**.** Let $\mathbf{r}(t) = \langle 8t,\, 3\sin(2t),\, 3\cos(2t)\rangle$.

(1) Find the unit tangent vector.
(2) Find the unit normal vector.

Date: 18 November 2020.

Solution to Part 1. The tangent vector is $\mathbf{r}'(t) = \langle 8,\, 6\cos(2t),\, -6\sin(2t)\rangle$. To get the unit tangent vector, multiply by 1 divided by the length:

$$\|\mathbf{r}'(t)\| = \sqrt{64 + 36\cos^2(2t) + 36\sin^2(2t)} = \sqrt{100} = 10.$$

So

$$\mathbf{T}(t) = \left(\frac{1}{\|\mathbf{r}'(t)\|}\right)\mathbf{r}'(t) = \left\langle \tfrac{4}{5},\, \tfrac{3}{5}\cos(2t),\, -\tfrac{3}{5}\sin(2t)\right\rangle.$$

This completes the solution. □

Solution to Part 2. Differentiate $\mathbf{T}(t)$, getting

$$\mathbf{T}'(t) = \left\langle 0,\, -\tfrac{6}{5}\sin(2t),\, -\tfrac{6}{5}\cos(2t)\right\rangle.$$

We want the unit vector in the same direction. So calculate:

$$\|\mathbf{T}'(t)\| = \sqrt{\tfrac{36}{25}\sin^2(2t) + \tfrac{36}{25}\cos^2(2t)} = \tfrac{6}{5}$$

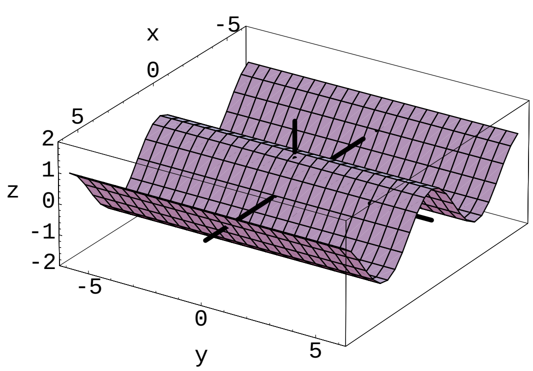
and

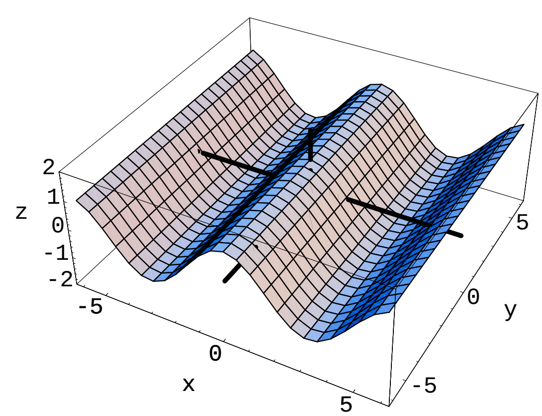
$$\mathbf{N}(t) = \left(\frac{1}{\|\mathbf{T}'(t)\|}\right)\mathbf{T}'(t) = \langle 0,\, -\sin(2t),\, -\cos(2t)\rangle.$$

This completes the solution. □

Problem 3 (15 points)**.** Sketch the graph of the function $f(x, y) = \cos(x)$.

Solution. The function depends only on x, so its graph is a generalized cylinder (in the sense of the book) oriented along the y-axis. Here are two views: from a traditional viewpoint (left), and from the viewpoint chosen by the computer (right):





□

Problem 4 (10 points)**.** Find $\lim_{(x,y)\to(0,0)} 2(x^2 + y^2)\cos\left(\dfrac{7x}{x^2 + y^2}\right)$, giving reasons for your answer, or explain why it does not exist.

Solution. Use the Squeeze Theorem. Clearly $-1 \le \cos\left(\dfrac{7x}{x^2 + y^2}\right) \le 1$, and $x^2 + y^2 \ge 0$, so

$$-2(x^2 + y^2) \le 2(x^2 + y^2)\cos\left(\frac{7x}{x^2 + y^2}\right) \le 2(x^2 + y^2)$$

for all $(x, y) \neq (0, 0)$. Also,

$$\lim_{(x,y)\to(0,0)} 2(x^2 + y^2) = \lim_{(x,y)\to(0,0)} [-2(x^2 + y^2)] = 0,$$

so $\lim_{(x,y)\to(0,0)} 2(x^2 + y^2) \cos\left(\frac{7x}{x^2 + y^2}\right) = 0.$ □

Problem 5 (15 points)**.** Let $f(x, t) = x^2 e^{-ct}$, where c is a constant. Find f_{ttxx}.

Solution. We have

$f_t(x, t) = -cx^2 e^{-ct}$, $\quad f_{tt}(x, t) = c^2 x^2 e^{-ct}$, $\quad f_{ttx}(x, t) = 2c^2 x e^{-ct}$, $\quad$ and $\quad f_{ttxx}(x, t) = 2c^2 e^{-ct}$.

This completes the solution. □

Problem 6 (10 points)**.** Let F, u, and v be differentiable functions of two variables, and let $W(s, t) = F(u(s, t), v(s, t))$. Suppose that

$$u(1, 0) = 2, \qquad u_s(1, 0) = -2, \qquad u_t(1, 0) = 6,$$

$$v(1, 0) = 3, \qquad v_s(1, 0) = -5, \qquad v_t(1, 0) = 4,$$

$$F(2, 3) = -7, \qquad F_u(2, 3) = -1, \qquad \text{and} \qquad F_v(2, 3) = 10.$$

Find $W_t(1, 0)$.

Solution. We have

$$\begin{aligned} W_t(1, 0) &= F_u(u(1, 0),\, v(1, 0))u_t(1, 0) + F_v(u(1, 0),\, v(1, 0))v_t(1, 0) \\ &= F_u(2, 3)u_t(1, 0) + F_v(2, 3)v_t(1, 0) = (-1)(6) + (10)(4) = 34. \end{aligned}$$

This completes the solution. □

Problem 7 (10 points)**.** A function g of two variables satisfies $g(3, 8) = 11$, $g_x(3, 8) = 2$, and $g_y(3, 8) = -6$. Use the linear approximation (tangent plane approximation) to estimate $g(3.1, 7.95)$.

Solution. The linear approximation is

$$g(x, y) \approx g(a, b) + g_x(a, b)(x - a) + g_y(a, b)(y - b).$$

(This is an *approximation*, not an equality, and therefore it is *not* correct to write "=" here.)

Take $a = 3$, $b = 8$, $x = 3.1$, and $y = 7.95$. This gives

$$\begin{aligned} g(3.1, 7.95) &\approx g(3, 8) + g_x(3, 8)(3.1 - 3) + g_y(3, 8)(7.95 - 8) \\ &= 11 + (2)(0.1) + (-6)(-0.05) = 11.5. \end{aligned}$$

Thus, $g(3.1, 7.95) \approx 11.5$. (Again, it is *not* correct to write "=" here.) □

Problem 8 (10 points)**.** Find the maximum rate of change of the function $f(r, s, t) = r^2 s^3 t^4$ at the point $(1, 1, 1)$, and the direction in which it occurs.

Solution. The direction is that of $\mathbf{grad}(f)(1, 1, 1)$. So calculate

$$\mathbf{grad}(f)(r, s, t) = \langle f_r(r, s, t),\, f_s(r, s, t),\, f_t(r, s, t)\rangle = \left\langle 2rs^3t^4,\, 3r^2s^2t^4,\, 4r^2s^3t^3 \right\rangle,$$

giving $\mathbf{grad}(f)(1,1,1) = \langle 2,3,4\rangle$. This vector points in the right direction, but isn't a unit vector. Our book's convention is that one should give a unit vector, so we multiply by

$$\frac{1}{\|\mathbf{grad}(f)(1,1,1)\|} = \frac{2}{\sqrt{29}},$$

getting the unit vector $\left\langle \frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}} \right\rangle$. For the rate, we take $\|\mathbf{grad}(f)(1,1,1)\| = \sqrt{29}$. □

Additional problems.

Problem 9 (10 points/part)**.** Let f be a scalar function which satisfies $f'(t) = \sin(t^3)$, let $\mathbf{u}$ be a vector function which satisfies $\mathbf{u}'(t) = -t^6\mathbf{u}(t)$, and let $\mathbf{v}$ be a vector function which satisfies $\mathbf{v}'(t) = -2\mathbf{v}(t) + \mathbf{i}$.

(1) Find $\dfrac{d}{dt}(f(t)\mathbf{u}(t))$. (Your answer might involve $f(t)$ and $\mathbf{u}(t)$.) Be sure to simplify your answer.
(2) Find $\dfrac{d}{dt}(\mathbf{u}(t)\cdot\mathbf{v}(t))$. (Your answer might involve $\mathbf{u}(t)$ and $\mathbf{v}(t)$.) Be sure to simplify your answer.
(3) Find $\dfrac{d}{dt}(\mathbf{u}(t)\times\mathbf{v}(t))$. (Your answer might involve $\mathbf{u}(t)$ and $\mathbf{v}(t)$.) Be sure to simplify your answer.
(4) Find $\dfrac{d}{dt}(\mathbf{u}(f(t)))$. (Your answer might involve $\mathbf{u}(t)$ and $f(t)$.) Be sure to simplify your answer.

Solution to Part 1. Use the product rule for scalar multiplication and substitute the given information:

$$\frac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t) = \sin(t^3)\mathbf{u}(t) + f(t)(-t^6\mathbf{u}(t)) = (\sin(t^3) - t^6 f(t))\mathbf{u}(t).$$

(The simplification is *required.*) □

Solution to Part 2. Use the product rule for the dot product and substitute the given information:

$$\begin{aligned}\frac{d}{dt}(\mathbf{u}(t)\cdot\mathbf{v}(t)) &= \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t) = (-t^6\mathbf{u}(t))\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot(-2\mathbf{v}(t)+\mathbf{i})\\ &= -t^6\mathbf{u}(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot(-2\mathbf{v}(t)+\mathbf{i}) = -(t^6+2)\mathbf{u}(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{i}.\end{aligned}$$

(The simplification is *required.*) □

Solution to Part 3. Use the product rule for the cross product and substitute the given information:

$$\begin{aligned}\frac{d}{dt}(\mathbf{u}(t)\times\mathbf{v}(t)) &= \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t) = (-t^6\mathbf{u}(t))\times\mathbf{v}(t) + \mathbf{u}(t)\times(-2\mathbf{v}(t)+\mathbf{i})\\ &= -t^6\mathbf{u}(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times(-2\mathbf{v}(t)+\mathbf{i}) = -(t^6+2)\mathbf{u}(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{i}.\end{aligned}$$

(The simplification is *required.*) □

Solution to Part 4. Use the chain rule for vector valued functions and substitute the given information:

$$\frac{d}{dt}(\mathbf{u}(f(t))) = f'(t)\mathbf{u}'(f(t)) = \sin(t^3)(-[f(t)]^6\mathbf{u}(f(t))) = -[f(t)]^6\sin(t^3)\mathbf{u}(f(t)).$$

(The simplification is *required.*) □

Problem 10 (9 points)**.** Let $\mathbf{v}$ be a vector function which satisfies $\mathbf{v}'(t) = -t\mathbf{v}(t) + 3\mathbf{k}$. Let $\mathbf{r}(t)$ be the vector valued function $\mathbf{r}(t) = \mathbf{v}(t^7)$. Find $\mathbf{r}'(t)$.

Solution. Use the chain rule for vector valued functions and substitute the given information:

$$\mathbf{r}'(t) = 7t^6\mathbf{v}'(t^7) = 7t^6(-t^7\mathbf{v}(t^7) + 3\mathbf{k}) = -7t^{13}\mathbf{v}(t^7) + 21t^6\mathbf{k}.$$

This completes the solution. □

Problem 11 (8 points)**.** Let $q(x,y,z) = \sec(2x^3 + 7y^2)z + e^{x^2z} + \dfrac{1}{e}$. Find $D_2q(x,y,z)$.

Solution. The expressions $2x^3$, z, e^{x^2z}, and $\dfrac{1}{e}$ are all to be treated as constants. Thus, using the chain rule,

$$q_y(x,y,z) = \sec(2x^3 + 7y^2)\tan(2x^3 + 7y^2)z \cdot 14y = 14yz\sec(2x^3 + 7y^2)\tan(2x^3 + 7y^2).$$

This completes the solution. □

Problem 12 (12 points)**.** Define a function F by $F(t) = \int_0^t e^{-s^2}\,ds$. Set $f(x,y) = F(7x - y^2)$. Find the first order partial derivatives of f.

Solution. We have $F'(t) = e^{-t^2}$ by the Fundamental Theorem of Calculus. So use the chain rule:

$$f_x(x,y) = F'(7x-y^2)\cdot 7 = 7e^{-(7x-y^2)^2} \qquad \text{and} \qquad f_y(x,y) = F'(7x-y^2)\cdot(-2y) = -2ye^{-(7x-y^2)^2}.$$

This completes the solution. □

Problem 13 (15 points)**.** Find the linear approximation to the function $h(x,y) = \sqrt{x^4 + y^2}$ at the point $(2,3)$, and use it to estimate the number $\sqrt{(2.05)^4 + (2.9)^2}$.

Solution. We have

$$h_x(x,y) = \tfrac{1}{2}(x^4+y^2)^{-1/2}\cdot 4x^3 = \frac{2x^3}{\sqrt{x^4+y^2}} \qquad \text{and} \qquad h_y(x,y) = \tfrac{1}{2}(x^4+y^2)^{-1/2}\cdot 2y = \frac{y}{\sqrt{x^4+y^2}}.$$

Therefore

$$h_x(2,3) = \frac{16}{\sqrt{25}} = \frac{16}{5} \qquad \text{and} \qquad h_y(2,3) = \frac{3}{\sqrt{25}} = \frac{3}{5}.$$

The linear approximation is

$$h(x,y) \approx h(a,b) + h_x(a,b)(x-a) + h_y(a,b)(y-b).$$

(This is an *approximation*, not an equality, and therefore it is *not* correct to write "=" here.)

Take $a = 2$, $b = 3$, $x = 2.05$, and $y = 2.9$. This gives

$$h(2.05, 2.98) \approx h(2,3) + h_x(2,3)(2.05-2) + h_y(2,3)(2.9-3) = 5 + \tfrac{16}{5}(0.05) + \tfrac{3}{5}(-0.1) = 5.1.$$

Thus, $h(2.05, 2.98) \approx 5.1$. (Again, it is *not* correct to write "=" here.) □

Problem 14 (15 points)**.** Find the linearization of the function $w(x,y) = \sqrt{6x + e^{4y}}$ at the point $(4, 0)$.

Solution. The linearization is

$$L(x,y) = w(a,b) + w_x(a,b)(x-a) + w_y(a,b)(y-b).$$

Take $a = 4$ and $b = 0$. We have $w(4,0) = \sqrt{6 \cdot 4 + 1} = 5$. Also,

$$w_x(x,y) = \tfrac{1}{2}(6x + e^{4y})^{-1/2} \cdot 6 = \frac{3}{\sqrt{6x + e^{4y}}}$$

and

$$w_y(x,y) = \tfrac{1}{2}(6x + e^{4y})^{-1/2} \cdot 4e^{4y} = \frac{2e^{4y}}{\sqrt{6x + e^{4y}}}.$$

Therefore

$$w_x(4,0) = \frac{3}{\sqrt{25}} = \frac{3}{5} \qquad \text{and} \qquad w_y(4,0) = \frac{2}{\sqrt{25}} = \frac{2}{5}.$$

The linearization is therefore

$$L(x,y) = 5 + \tfrac{3}{5}(x-4) + \tfrac{2}{5}y.$$

(We thus get

$$w(x,y) \approx L(x,y) = 5 + \tfrac{3}{5}(x-4) + \tfrac{2}{5}y.$$

This is an *approximation*, not an equality. So it is *not* correct to write "~~$w(x,y) = L(x,y)$~~" here.) □

Problem 15 (16 points)**.** Let $f(x,y) = \ln(3x + 7y)$. Find all second order partial derivatives of f.

Solution. We have

$$f_x(x,y) = \frac{3}{3x+7y} \qquad \text{and} \qquad f_y(x,y) = \frac{7}{3x+7y}.$$

Therefore

$$f_{xx}(x,y) = \frac{\partial}{\partial x}\left(\frac{3}{3x+7y}\right) = -\frac{9}{(3x+7y)^2},$$

$$f_{xy}(x,y) = \frac{\partial}{\partial y}\left(\frac{3}{3x+7y}\right) = -\frac{21}{(3x+7y)^2},$$

$$f_{yx}(x,y) = \frac{\partial}{\partial x}\left(\frac{7}{3x+7y}\right) = -\frac{21}{(3x+7y)^2},$$

and

$$f_{yy}(x,y) = \frac{\partial}{\partial y}\left(\frac{7}{3x+7y}\right) = -\frac{49}{(3x+7y)^2}.$$

This completes the solution. □

Problem 16 (10 points)**.** Let f be a differentiable function of three variables. Let x, y, and z be differentiable functions of two variables, and set $w(s,t) = f(x(s,t),\, y(s,t),\, z(s,t))$. Find $w_s(s,t)$ in terms of the functions f, x, y, and z and their partial derivatives.

Solution. Use the chain rule:

$$\begin{aligned} w_s(s,t) &= f_x(x(s,t),\, y(s,t),\, z(s,t))x_s(s,t) + f_y(x(s,t),\, y(s,t),\, z(s,t))y_s(s,t) \\ &\quad + f_z(x(s,t),\, y(s,t),\, z(s,t))z_s(s,t). \end{aligned}$$

This completes the solution. □

Problem 17 (10 points)**.** Let f be a differentiable function of two variables. Let $x(s,t) = e^s\cos(t)$ and $y(s,t) = e^s\sin(t)$, and set $u(s,t) = f(x(s,t),\, y(s,t))$. Find $u_t(s,t)$ in terms of s, t, and the function f and its partial derivatives.

Solution. Use the chain rule:

$$\begin{aligned} u_t(s,t) &= f_x(x(s,t),\, y(s,t))x_t(s,t) + f_y(x(s,t),\, y(s,t))y_t(s,t) \\ &= f_x(e^s\cos(t),\, e^s\sin(t))(-e^s\sin(t)) + f_y(e^s\cos(t),\, e^s\sin(t))e^s\cos(t) \\ &= -e^s\sin(t)f_x(e^s\cos(t),\, e^s\sin(t)) + e^s\cos(t)f_y(e^s\cos(t),\, e^s\sin(t)). \end{aligned}$$

This completes the solution. □

Problem 18 (10 points)**.** Let f be a differentiable function of two variables. (In the notation for partial derivatives, call them x and y.) Let $g(r,s) = f(3r - s - 1,\, s^2 - 4r)$. Suppose

$$f(0,0) = 3, \qquad g(0,0) = 6, \qquad f_x(0,0) = 4, \qquad \text{and} \qquad f_y(0,0) = 8,$$

and

$$f(1,2) = 6, \qquad g(1,2) = 3, \qquad f_x(1,2) = 2, \qquad \text{and} \qquad f_y(1,2) = 5.$$

Find $g_s(1,2)$.

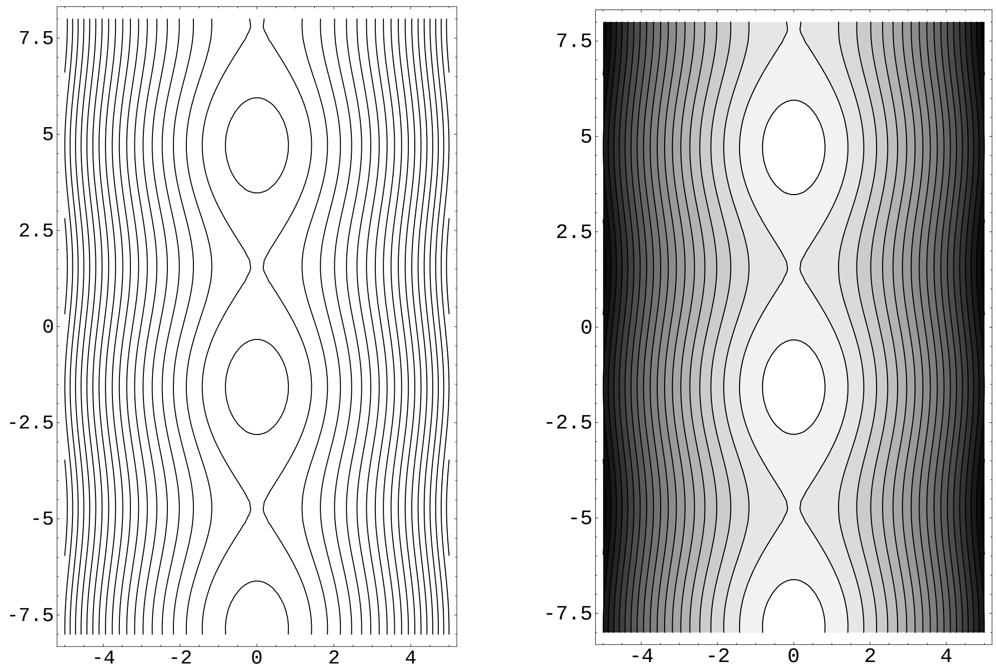
Solution. Set $x(r,s) = 3r - s - 1$ and $y(r,s) = s^2 - 4r$. Then $x_s(r,s) = -1$ and $y_s(r,s) = 2s$. So

$$\begin{aligned} g_s(1,2) &= f_x(x(1,2),\, y(1,2))x_s(1,2) + f_y(x(1,2),\, y(1,2))y_s(1,2) \\ &= f_x(0,0)x_s(1,2) + f_y(0,0)y_s(1,2) = (4)(-1) + (8)(4) = 28. \end{aligned}$$

This completes the solution. □

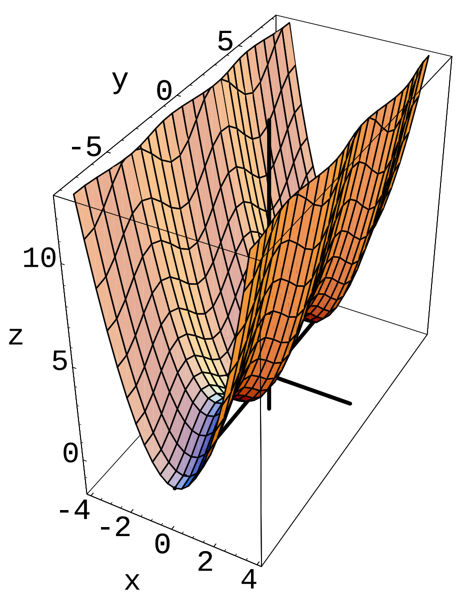
Problem 19 (15 points)**.** Draw a contour map of the function $f(x,y) = x^2+1+\sin(y)$, showing level curves for at least three values in the range.

Solution. The graph of the function looks like a parabolic valley along the y-axis, with an oscillation in the y direction coming from the term $\sin(y)$. Here are two contour maps: unshaded (left), and shaded with higher elevations darker (right):



This completes the solution. □

For comparison, here is a graph over a slightly narrower region (*not* asked for in the problem):



Problem 20 (10 points)**.** Find $\lim_{(x,y)\to(6,3)} xy\cos(x-2y)$, giving reasons for your answer, or explain why it does not exist.

Solution. The function $f(x,y) = xy\cos(x-2y)$ is defined and continuous on all of $\mathbb{R}^2$, so

$$\lim_{(x,y)\to(6,3)} xy\cos(x-2y) = f(6,3) = 18\cos(0) = 18.$$

(The simplification is *necessary*.) □

Problem 21 (10 points)**.** Find $\lim_{(x,y)\to(0,0)} \dfrac{2}{x^2+y^4}$, giving reasons for your answer, or explain why it does not exist.

Solution. The limit does not exist because the function gets arbitrarily large as $(x,y)\to(0,0)$. Formally, $\lim_{(x,y)\to(0,0)} \dfrac{2}{x^2+y^4} = \infty$, but ∞ is not a number. So the limit does not exist. □

Problem 22 (10 points)**.** Find $\lim_{(x,y)\to(0,0)} \cos\left(\dfrac{7x}{x^2+y^2}\right)$, giving reasons for your answer, or explain why it does not exist.

Solution. Consider the values on the line $y=0$ (the x-axis). For $y=0$, we have

$$\cos\left(\frac{7x}{x^2+y^2}\right) = \cos\left(\frac{7x}{x^2}\right) = \cos\left(\frac{7}{x}\right).$$

Since $\lim_{x\to 0} \cos\left(\dfrac{7}{x}\right)$ does not exist (the function oscillates ever more rapidly between -1 and 1), the limit $\lim_{(x,y)\to(0,0)} \cos\left(\dfrac{7x}{x^2+y^2}\right)$ cannot exist either. □

Problem 23 (10 points)**.** Find $\lim_{(x,y)\to(6,3)} \dfrac{x(y-3)}{(x-2)(y^2-2y-3)}$, giving reasons for your answer, or explain why it does not exist.

Solution.

$$\begin{aligned}\lim_{(x,y)\to(6,3)} \frac{x(y-3)}{(x-2)(y^2-2y-3)} &= \lim_{(x,y)\to(6,3)} \frac{x(y-3)}{(x-2)(y+1)(y-3)} \\ &= \lim_{(x,y)\to(6,3)} \frac{x}{(x-2)(y+1)} = \frac{6}{(4)(4)} = \frac{3}{8}.\end{aligned}$$

This completes the solution. □

Problem 24 (12 points)**.** Find the directional derivative of $f(x,y) = x\sin(xy)$ at the point $(2,0)$ in the direction determined by the angle $\pi/3$.

Solution. First,

$$f_x(x,y) = \sin(xy) + xy\cos(xy) \qquad \text{and} \qquad f_y(x,y) = x^2\cos(xy).$$

Therefore

$$f_x(2,0) = 0 \qquad \text{and} \qquad f_y(2,0) = 4.$$

Next, find the unit vector in the required direction; it is $\mathbf{u} = \left\langle \frac{1}{2}, \sqrt{3}/2 \right\rangle$. So

$$D_{\mathbf{u}} f(2,0) = f_x(2,0)\cdot\tfrac{1}{2} + f_y(2,0)\sqrt{3}/2 = 2\sqrt{3}.$$

This completes the solution. □

Problem 25 (points per part as specified)**.** Let $f(x,y,z) = \sqrt{x+yz}$.

(1) (8 points.) Find $\mathbf{grad}(f)$.
(2) (4 points.) Evaluate $\mathbf{grad}(f)$ at the point $P = (1,3,1)$.
(3) Find the rate of change of f at the point P in the direction $\left\langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7} \right\rangle$.

Solution to Part 1.

$$\begin{aligned}\mathbf{grad}(f)(x,y,z) &= \langle f_x(x,y,z),\ f_y(x,y,z),\ f_z(x,y,z)\rangle \\ &= \left\langle \tfrac{1}{2}(x+yz)^{-1/2},\ \tfrac{1}{2}z(x+yz)^{-1/2},\ \tfrac{1}{2}y(x+yz)^{-1/2} \right\rangle.\end{aligned}$$

This completes the solution. □

Solution to Part 2. $\mathbf{grad}(f)(1,3,1) = \left\langle \frac{1}{4}, \frac{1}{4}, \frac{3}{4} \right\rangle$. □

Solution to Part 3. Set $\mathbf{u} = \left\langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7} \right\rangle$. This is a unit vector. Then we want

$$D_{\mathbf{u}} f(1,3,1) = \mathbf{grad}(f)(1,3,1) \cdot \mathbf{u} = \tfrac{23}{28}.$$

This completes the solution. □

Problem 26 (12 points)**.** Find an equation for the tangent plane to the surface $x - z = 4\arctan(yz)$ at the point $(1+\pi,\ 1,\ 1)$.

Solution. Set $f(x,y,z) = x - z - 4\arctan(yz)$, so that the surface in question is the level surface $f(x,y,z) = 0$. For the normal vector to the plane, we take $\mathbf{n} = \mathbf{grad}(f)(1+\pi,\ 1,\ 1)$. So begin by calculating

$$\mathbf{grad}(f)(x,y,z) = \langle f_x(x,y,z),\ f_y(x,y,z),\ f_z(x,y,z)\rangle = \left\langle 1,\ -\frac{4z}{1+y^2z^2},\ -1-\frac{4y}{1+y^2z^2} \right\rangle$$

and

$$\mathbf{grad}(f)(1+\pi,\ 1,\ 1) = \langle 1,\ -2,\ -3 \rangle.$$

The plane goes through the point $(1+\pi,\ 1,\ 1)$, so we get the equation

$$\langle 1,\ -2,\ -3\rangle \cdot (\langle x,y,z\rangle - \langle 1+\pi,\ 1,\ 1\rangle) = 0,$$

that is, $x - (1+\pi) - 2(y-1) - 3(z-1) = 0$. We simplify this to $x - 2y - 3z + 4 - \pi = 0$. □

Alternate solution (outline). Set $g(y,z) = z + 4\arctan(yz)$. Find the tangent plane to the graph of $x = g(y,z)$ using the methods of Section 14.4 of the book. Be sure not to mix up the variables. □

Problem 27 (12 points)**.** Find parametric equations for the normal line to the surface $x - z = 4\arctan(yz)$ at the point $(1+\pi,\ 1,\ 1)$.

Solution. Set $f(x,y,z) = x - z - 4\arctan(yz)$, so that the surface in question is the level surface $f(x,y,z) = 0$. For the direction vector for the line, we take $\mathbf{v} = \mathbf{grad}(f)(1+\pi,\ 1,\ 1)$ (which is a normal vector to the surface). So begin by calculating

$$\mathbf{grad}(f)(x,y,z) = \langle f_x(x,y,z),\ f_y(x,y,z),\ f_z(x,y,z)\rangle = \left\langle 1,\ -\frac{4z}{1+y^2z^2},\ -1-\frac{4y}{1+y^2z^2} \right\rangle$$

and

$$\mathbf{grad}(f)(1+\pi,\ 1,\ 1) = \langle 1,\ -2,\ -3 \rangle.$$

The line goes through the point $(1+\pi,\ 1,\ 1)$, so we get the equation

$$\mathbf{r} = \langle 1+\pi,\ 1,\ 1\rangle + t\langle 1,\ -2,\ -3\rangle,$$

which gives the parametric equations

$$x(t) = 1+\pi+t, \qquad y(t) = 1-2t, \qquad z(t) = 1-3t.$$

This completes the solution. □

Problem 28 (12 points)**.** Explain why there is no function f such that $f_x(x,y) = 3x\sin(y)$ and $f_y(x,y) = 2x^2\cos(y)$.

Solution. Suppose f were such a function. Then we would have

$$f_{xy}(x,y) = \frac{\partial}{\partial y}(3x\sin(y)) = 3x\cos(y) \qquad \text{and} \qquad f_{yx}(x,y) = \frac{\partial}{\partial x}(2x^2\cos(y)) = 4x\cos(y).$$

These are not equal (for example, they differ at $(x,y) = (1,0)$), and both are continuous, so this is a contradiction. □

Problem 29 (10 points)**.** Find an equation for the tangent plane to the surface $z = e^{x^2-y^2}$ at the point $(1,\ -1,\ 1)$.

Solution. The tangent plane is the graph of the linearization. Accordingly, we let $f(x,y) = e^{x^2-y^2}$ and calculate

$$f_x(x,y) = 2xe^{x^2-y^2} \qquad \text{and} \qquad f_y(x,y) = -2ye^{x^2-y^2},$$

so $f_x(1,\ -1) = 2$ and $f_y(1,\ -1) = 2$. Therefore the equation is $z = 1 + 2(x-1) + 2(y+1)$, which simplifies to $z = 2x + 2y + 1$.

Alternate solution: Set $g(x,y,z) = z - e^{x^2-y^2}$, so that the surface in question is the level surface $g(x,y,z) = 0$. For the normal vector to the plane, we take $\mathbf{n} = \mathbf{grad}(g)(1,\ -1,\ 1)$. So begin by calculating

$$\mathbf{grad}(g)(x,y,z) = \langle g_x(x,y,z),\ g_y(x,y,z),\ g_z(x,y,z)\rangle = \left\langle -2xe^{x^2-y^2},\ 2ye^{x^2-y^2},\ 1\right\rangle$$

and

$$\mathbf{grad}(g)(1,\ -1,\ 1) = \langle -2,\ -2,\ 1\rangle.$$

The plane goes through the point $(1,\ -1,\ 1)$, so we get the equation

$$\langle -2,\ -2,\ 1\rangle \cdot (\langle x,y,z\rangle - \langle 1,\ -1,\ 1\rangle) = 0,$$

that is, $-2(x-1) - 2(y+1) + (z-1) = 0$. We simplify this to $-2x - 2y + z - 1 = 0$. □